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ON EXTENDED SPECTRUM OF A COMPOSITION OPERATOR ON SEQUENCE SPACES

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Abstract. In this paper, we investigate the spectral properties of a composition operator $C_\alpha : \ell^p \to \ell^p$, where $1 \leq p < \infty$ and a null sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$, defined by $C_\alpha x = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots)$. In particular, we prove that the point spectrum, the approximate point spectrum as well as the spectrum of C_α all coincide and is the singleton set containing 0. However, the extended spectrum of C_α turns out to be the complement of the closed unit disc of the complex plane.

1. Introduction

Let X be an arbitrary Banach space and $\mathcal{L}(X)$ be the space of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, the resolvent set of T, $\rho(T)$, is given by $\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}$ and its spectrum $\delta(T) := \mathbb{C} \setminus \rho(T)$. The set of all eigenvalues of T is called its point spectrum denoted by $\delta_p(T)$, while the approximate point spectrum of T, $\delta_{ap}(T)$, is the set of all $\lambda \in \mathbb{C}$ for which there exist unit vectors $x_n \in X$ such that $\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$. The spectral radius of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \delta(T)\}$. For a detailed theory on spectra we refer to [3, 6, 13, 19].

A complex number $\lambda \in \mathbb{C}$ is called an extended eigenvalue of an operator $T \in \mathcal{L}(X)$ if there exists a nonzero operator $A \in \mathcal{L}(X)$ such that $AT = \lambda TA$. The set of all extended eigenvalues of T will be denoted by $\delta_{Ext}(T)$ and is called the extended spectrum of T, while such an operator $A \in \mathcal{L}(X)$ is called the extended eigenoperator associated with the extended eigenvalue λ of T. It is well known that $\delta_{Ext}(T)$ is a nonempty and closed subset of $\mathbb{C}$. We refer to [7, 10, 14, 15, 16] for a detailed theory on extended spectra.

The study of the extended spectrum of bounded operators has been considered in different settings, see [1, 2, 4, 7, 9, 11] and references therein. Biswas et. al. [2] considered the Volterra operator, while Lacruz et. al. [18] determined the extended spectrum for discrete, finite and continuous Cesàro operators. In [9, 16, 12], interesting results on extended spectrum for shift operators are given on the sequence space ℓ^2. It is important to note that sequence spaces still remain

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a useful setting for spectral analysis of a variety of operators, see for instance the recent work by Nuray [5]. In this paper, we consider a composition of a right shift operator and a multiplication operator on the general sequence space ℓ^p, $1 \leq p \leq \infty$, and investigate its spectral properties. In particular, we determine explicitly the extended spectrum of the resulting composition operator.

2. Preliminary Results

Let ℓ^p, $1 \leq p < \infty$ denotes the space of p-summable sequences of real or complex numbers while ℓ^∞ denotes the corresponding space of bounded sequences of real or complex numbers. It is well known that ℓ^p, $1 \leq p \leq \infty$ are Banach spaces with respect to the norms $\|\cdot\|_p$ given by

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \qquad \text{for all } x = (x_k)_{k=1}^{\infty} \in \ell^p,\ 1 \leq p < \infty,$$

while

$$\|x\|_\infty = \max_{k\in\mathbb{N}} |x_k| \qquad \text{for all } x = (x_k)_{k=1}^{\infty} \in \ell^\infty.$$

The Unilateral right shift operator $S : \ell^p \to \ell^p$ is defined by

$$Sx = (0, x_1, x_2, \ldots) \quad \text{for all } x = (x_k)_{k=1}^{\infty} \in \ell^p. \tag{2.1}$$

For an arbitrary null sequence $(\alpha_k)_{k\in\mathbb{N}}$ of real or complex numbers, define a Multiplication operator $M_\alpha : \ell^p \to \ell^p$ by

$$M_\alpha x = (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots) \quad \text{for all } x = (x_k)_{k\in\mathbb{N}} \in \ell^p. \tag{2.2}$$

We then define the composition of S and M_α on ℓ^p by the operator $C_\alpha := S \circ M_\alpha : \ell^p \to \ell^p$. Specifically,

$$C_\alpha x = (0, \alpha_1 x_1, \alpha_2 x_2, \ldots) \quad \text{for all } x = (x_k)_{k\in\mathbb{N}} \in \ell^p. \tag{2.3}$$

For an arbitrary $0 \neq \lambda \in \mathbb{C}$, we define the diagonal operators D_λ and $D_{\frac{1}{\lambda}}$ on ℓ^p as follows:

$$D_\lambda = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda^2 & \\ & & & \ddots \end{bmatrix}$$

and

$$D_{\frac{1}{\lambda}} = \begin{bmatrix} 1 & & & \\ & \frac{1}{\lambda} & & \\ & & \frac{1}{\lambda^2} & \\ & & & \ddots \end{bmatrix}$$

It can be easily verified that $D_{\frac{1}{\lambda}} : \ell^p \to \ell^p$ whenever $|\lambda| \geq 1$ while $D_\lambda : \ell^p \to \ell^p$ whenever $|\lambda| < 1$. We end this section by detailing the spectral properties of the operators S, M_α and C_α as defined on the ℓ^p spaces.

Theorem 2.1. *Let $S : \ell^p \to \ell^p$ be the unilateral right shift operator defined by equation (2.1). Then:*

(1) *S is an isometry with $\|S\| = 1$*
(2) *$\delta_p(S) = \emptyset$, that is, S has no eigenvalues*
(3) *$\delta(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, which is a closed unit disk $\overline{\mathbb{D}}$*
(4) *$\delta_{ap}(S) = \partial\mathbb{D}$, where $\partial\mathbb{D}$ is the unit circle.*
(5) *$r(S) = 1$.*

Proof. To prove (1), for all $x \in \ell^p, 1 \leq p < \infty$, we have

$$\begin{aligned} \|Sx\|_p^p &= |0|^p + |x_1|^p + |x_3|^p + \ldots \\ &= \Sigma_{k=1}^{\infty} |x_k|^p \\ &= \|x\|_p^p. \end{aligned}$$

That is, $\|Sx\|_p = \|x\|_p$. Now,

$$\begin{aligned} \|Sx\|_\infty &= \max_{k \in \mathbb{N}} |x_k| \\ &= \|x\|_\infty. \end{aligned}$$

Taking supremum over all $x \in \ell^p$, $1 \leq p \leq \infty$, with $\|x\|_p = 1$, we obtain the desired result. To prove (2), we have that $\lambda \in \delta_p(S)$ if and only if $Sx = \lambda x$ for some nonzero $x \in \ell^p$. This is equivalent to $(0, x_1, x_2, \ldots) = \lambda(x_1, x_2, x_3, \ldots)$. If $\lambda = 0$, then clearly $x = 0$ and if $\lambda \neq 0$, then $x = 0$ as well. Hence there are no eigenvalues. Next we note that S is not surjective since $(1, 0, 0, \ldots)$ does not belong to the range of S. Therefore by using (1), S is a non-invertible isometry. It then follows immediately that $\delta(S) = \nabla(0, 1)$ and $\delta_{ap}(S) = \partial\nabla(0, 1)$. This proves assertions (3) and (4). Assertion (5) follows from the definition of the spectral radius and assertion (3). □

Theorem 2.2. *Let $M_\alpha : \ell^p \to \ell^p$ be the multiplication operator defined as in (2.2). Then*

(1) $\|M_\alpha\| = \max_{n \in \mathbb{N}} |\alpha_n|$
(2) $\delta_p(M_\alpha) = \{\alpha_n : n \in \mathbb{N}\}$
(3) $\delta(M_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}} = \{0\} \cup \{\alpha_n : n \in \mathbb{N}\}$
(4) $\delta_{ap}(M_\alpha) = \delta(M_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}}$
(5) $r(M_\alpha) = \max_{n \in \mathbb{N}} |\alpha_n|$.

Proof. To prove (1), we have for all $x \in \ell^p$,

$$\begin{aligned} \|M_\alpha x\|_p^p &= \sum_{n=1}^{\infty} |\alpha_n|_p^p |x_n|_p^p \\ &\leq \max_{n \in \mathbb{N}} |\alpha_n|_p^p \|x\|_p^p. \end{aligned}$$

Taking the p^{th}-root on both sides yields $\|M_\alpha\|_p \leq \max_{n \in \mathbb{N}} |\alpha_n| \|x\|_p$. The result now follows by taking supremum over all $x \in \text{Ball}(\ell^p)$. On the other hand, let $e_n = (0, 0, ..., 0, 1, 0, ...)$ the unit vector having 1 at the n^{th} position and zero elsewhere. Then $e_n \in \ell^p$ with $\|e_n\|_p = 1$ and so $\|M_\alpha\| \geq \|M_\alpha e_n\|_p = |\alpha_n|$. Taking maximum over all $n \in \mathbb{N}$ completes the proof of (1). Now, $\lambda \in \delta_p(M_\alpha)$ is equivalent to $M_\alpha x = \lambda x$ for some $0 \neq x \in \ell^p$ or $(\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots) = \lambda(x_1, x_2, x_3, \ldots)$. If $\lambda \neq 0$, then $\alpha_1 = \lambda, \alpha_2 = \lambda, \ldots$ and therefore $\{\alpha_n : n \in$

$\mathbb{N}\} \subseteq \delta_p(M_\alpha)$. If $\lambda = 0$, then $x_1 = 0 = x_2 = x_3 = \ldots$ and so $\lambda = 0$ is not an eigenvalue. Therefore $\delta_p(M_\alpha) = \{\alpha_n : n \in \mathbb{N}\}$, as desired. Following assertion 2, we have that $\overline{\delta_p(M_\alpha)} \subseteq \delta(M_\alpha)$ since the spectrum is always closed, where $\overline{\delta_p(M_\alpha)} = \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$. We now prove that $\delta(M_\alpha) \subseteq \overline{\delta_p(M_\alpha)}$. Let $\lambda \in \mathbb{C}, \lambda \neq 0$ and $\lambda \notin \delta_p(M_\alpha)$. This implies that $\lambda \in \rho(M_\alpha)$ which is equivalent to $\lambda I - M_\alpha$ invertible, that is, $\lambda I - M_\alpha : \ell^p \to \ell^p$ is bijective. Therefore for all $y \in \ell^p$, there exists a unique $x \in \ell^p$ such that $(\lambda I - M_\alpha)x = y$. Taking $x = \left(\frac{y_j}{\lambda - \alpha_j}\right)_{j \in \mathbb{N}}$, it follows that $x \in \ell^p$ and therefore $\delta(M_\alpha) = \overline{\{\alpha_n : n \in \mathbb{N}\}}$ which proves (4). To prove (5), we know that $\delta_p(M_\alpha) \subseteq \delta_{ap}(M_\alpha) \subseteq \delta(M_\alpha)$, that is, $\{\alpha_n : n \in \mathbb{N}\} \subseteq \delta_{ap}(M_\alpha) \subseteq \{\alpha_n : n \in \mathbb{N}\} \cup \{0\}$. Since $\delta_{ap}(M_\alpha)$ is closed, it follows that $\delta_{ap}(M_\alpha) = \delta(M_\alpha)$. This therefore implies that the spectral radius is, $r(M_\alpha) = \max_{n \in \mathbb{N}} \mid \alpha_n \mid$, which proves (6). □

Theorem 2.3. *Let $C_\alpha : \ell^p \to \ell^p$, be the composition operator defined by (2.3). Then*

(1) $\|C_\alpha\| = \max_{n \in \mathbb{N}} |\alpha_n|$
(2) $\delta_p(C_\alpha) = \{0\}$
(3) $\delta(C_\alpha) = \{0\}$
(4) $\delta_{ap}(C_\alpha) = \{0\}$
(5) $r(C_\alpha) = 0$.

Proof. For every $x \in \ell^p$ and $(\alpha_n)_{n \in \mathbb{N}} \subseteq \mathbb{F}$ such that $\alpha_n \to 0$, we have

$$\begin{aligned} \|C_\alpha x\|_p^p &= \sum_{n=1}^{\infty} |\alpha_n x_n|^p \\ &\leq \max_{n \in \mathbb{N}} |\alpha_n|^p \sum_{n=1}^{\infty} |x_n|^p \\ &= \max_{n \in \mathbb{N}} |\alpha_n|^p \|x\|_p^p. \end{aligned}$$

Therefore,

$$\|C_\alpha\| \leq \max_{n \in \mathbb{N}} |\alpha_n|.$$

To prove the reverse inequality, take $x = e^n = (0, 0, ..., 1, 0, 0, ...)$, a unit vector having 1 at the nth position and zero elsewhere. Then,

$$\|C_\alpha\| \geq \|C_\alpha e^n\|_p = |\alpha_n| \implies \|C_\alpha\| \geq \max_{n \in \mathbb{N}} |\alpha_n|.$$

Now, $\lambda \in \delta_p(C_\alpha) \Leftrightarrow C_\alpha x = \lambda x$ for some $0 \neq x \in \ell^p$. Equivalently, $(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots) = \lambda(x_1, x_2, x_3, \ldots)$. Therefore if $\lambda \neq 0$, then $0 = x_1 = x_2 = x_3 = \ldots$ and so $\lambda \neq 0$ is not an eigenvalue of C_α. For $\lambda = 0$, $\alpha_n = 0$ for some n implying that there exists some nonzero eigenvector. Therefore $\delta_p(C_\alpha) = \{0\}$, which proves (2). To prove (3),

$$\begin{aligned} C_\alpha^2(x) &= C_\alpha(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, ...) \\ &= (0, 0, \alpha_2 \alpha_1 x_1, \alpha_3 \alpha_2 x_2, ...). \end{aligned}$$

implying that,

$$\|C_\alpha^2\| = \sup_{n\in\mathbb{N}} |\alpha_n\alpha_{n+1}|.$$

Also

$$\begin{aligned} C_\alpha^3(x) &= C_\alpha(C_\alpha^2(x)) \\ &= C_\alpha(0,0,\alpha_2\alpha_1x_1,\alpha_3\alpha_2x_2,...) \\ &= (0,0,0,\alpha_3\alpha_2\alpha_1x_1,\alpha_4\alpha_3\alpha_2x_2,...). \end{aligned}$$

and thus,

$$\|C_\alpha^3\| = \sup_{n\in\mathbb{N}} |\alpha_n\alpha_{n+1}\alpha_{n+2}|.$$

Implying that in general, $\|C_\alpha^k\| = \sup_{n\in\mathbb{N}} |\alpha_n\alpha_{n+1}\alpha_{n+2}...\alpha_{n+k}|$.
Therefore $\lim_{k\to\infty} \|C_\alpha^k\|^{\frac{1}{k}} = 0$ and thus $r(C_\alpha) = 0$ proving (5). It therefore follows that $\delta(C_\alpha) = \{0\}$, which proves (3). Now, obviously $\delta_{ap}(C_\alpha) = \delta(C_\alpha) = \{0\}$, which completes the proof. □

3. Extended spectrum

We compute the extended spectrum for the composition operator C_α in comparison with the extended spectrum of the operators S and M_α. It turns out that there is no obvious relation between the point spectrum δ_p, the approximate point spectrum δ_{ap}, the spectrum δ as well as the extended spectrum of the operators S, M_α and C_α.

Theorem 3.1. *Let S, M_α and C_α be defined on ℓ^p as given by equations (2.1),(2.2) and (2.3) respectively. Then the following holds:*

1. $\delta_{Ext}(S) = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$
2. $\delta_{Ext}(M_\alpha) = \{1\}$
3. $\delta_{Ext}(C_\alpha) = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$.

Proof. Let $S : \ell^p \to \ell^p$ be the unilateral right shift operator defined by equation (2.1), and suppose that $|\lambda| \geq 1$ and $D_{\frac{1}{\lambda}}$ is the diagonal operator on ℓ^p so that $D_{\frac{1}{\lambda}}(x_1, x_2, x_3, ...) = (x_1, \frac{1}{\lambda}x_2, \frac{1}{\lambda^2}x_3, ...)$ then

$$\begin{aligned} SD_{\frac{1}{\lambda}}(x_1,x_2,x_3,...) &= S(x_1,\frac{1}{\lambda}x_2,\frac{1}{\lambda^2}x_3,...) \\ &= (0,x_1,\frac{1}{\lambda}x_2,\frac{1}{\lambda^2}x_3,...) \end{aligned}$$

and

$$\begin{aligned} \lambda D_{\frac{1}{\lambda}}S(x_1,x_2,x_3,...) &= \lambda D_{\frac{1}{\lambda}}(0,x_1,x_2,x_3,...) \\ &= \lambda(0,\frac{1}{\lambda}x_1,\frac{1}{\lambda^2}x_2,\frac{1}{\lambda^3}x_3,...) \\ &= (0,x_1,\frac{1}{\lambda}x_2,\frac{1}{\lambda^2}x_3,...). \end{aligned}$$

Therefore from the two above equations $SD_{\frac{1}{\lambda}} = \lambda D_{\frac{1}{\lambda}}S$. So, $\{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \subseteq \delta_{Ext}(S)$. Now we need to show that $\delta_{Ext}(S) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$, that is,

$\lambda \in \delta_{Ext}(S) \Rightarrow |\lambda| \geq 1$.
We use the proof by contrapositive. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < 1$ and consider the diagonal operator D_λ. Then

$$\begin{aligned} SD_\lambda(x_1, x_2, x_3, ...) &= S(x_1, \lambda x_2, \lambda^2 x_3, ...) \\ &= (0, x_1, \lambda x_2, \lambda^2 x_3, ...), \end{aligned}$$

and

$$\begin{aligned} \lambda D_\lambda S(x_1, x_2, x_3, ...) &= \lambda D_\lambda(0, x_1, x_2, x_3, ...) \\ &= \lambda(0, \lambda x_1, \lambda^2 x_2, \lambda^3 x_3) \\ &= (0, \lambda^2 x_1, \lambda^3 x_2, \lambda^4 x_3, ...). \end{aligned}$$

Thus $SD_\lambda \neq \lambda D_\lambda S$, implying that $\lambda \notin \delta_{Ext}(S)$. Therefore $\delta_{Ext}(S) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$, implying that

$$\delta_{Ext}(S) = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}.$$

This completes the proof of (1).
To prove (2), let $M_\alpha : \ell^p \to \ell^p$ be the multiplication operator defined as in (2.2), and suppose that $|\lambda| \geq 1$ and $D_{\frac{1}{\lambda}}$ is the diagonal operator on ℓ^p so that

$$D_{\frac{1}{\lambda}}(x_1, x_2, x_3, ...) = (x_1, \frac{1}{\lambda} x_2, \frac{1}{\lambda^2} x_3, ...).$$

Then

$$\begin{aligned} M_\alpha D_{\frac{1}{\lambda}}(x_1, x_2, x_3, ...) &= M_\alpha(x_1, \frac{1}{\lambda} x_2, \frac{1}{\lambda^2} x_3, \dots) \\ &= (\alpha_1 x_1, \alpha_2 \frac{1}{\lambda} x_2, \alpha_3 \frac{1}{\lambda^2} x_3, \dots), \end{aligned}$$

and

$$\begin{aligned} \lambda D_{\frac{1}{\lambda}} M_\alpha(x_1, x_2, x_3, ...) &= \lambda D_{\frac{1}{\lambda}}(\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, ...) \\ &= \lambda(\alpha_1 x_1, \alpha_2 \frac{1}{\lambda} x_2, \alpha_3 \frac{1}{\lambda^2} x_3, ...) \\ &= (\alpha_1 \lambda x_1, \alpha_2 x_2, \alpha_3 \frac{1}{\lambda} x_3, ...). \end{aligned}$$

Thus for $\lambda = 1, M_\alpha D_{\frac{1}{\lambda}} = \lambda D_{\frac{1}{\lambda}} M_\alpha$ and therefore $1 \in \delta_{Ext}(M_\alpha)$.
On the other hand, for $\lambda \neq 1, M_\alpha D_{\frac{1}{\lambda}} \neq \lambda D_{\frac{1}{\lambda}} M_\alpha$ and so $\{\lambda \in \mathbb{C} : |\lambda| > 1\} \nsubseteq \delta_{Ext}(M_\alpha)$. Using the proof by contrapositive, Suppose $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and let D_λ be the diagonal operator on ℓ^p. Then

$$\begin{aligned} M_\alpha D_\lambda(x_1, x_2, x_3, ...) &= M_\alpha(x_1, \lambda x_2, \lambda^2 x_3, ...) \\ &= (\alpha_1 x_1, \alpha_2 \lambda x_2, \alpha_3 \lambda^2 x_3, ...), \end{aligned}$$

and

$$\begin{aligned} \lambda D_\lambda M_\alpha(x_1, x_2, x_3, ...) &= \lambda D_\lambda(\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, ...) \\ &= \lambda(\alpha_1 x_1, \alpha_2 \lambda x_2, \alpha_3 \lambda x_2, \alpha_3 \lambda^2 x_3, ...) \\ &= (\lambda \alpha_1 x_1, \lambda^2 \alpha_2 x_2, \lambda^3 \alpha_3 x_3, ...). \end{aligned}$$

Thus for $|\lambda| < 1, M_\alpha D_\lambda \neq \lambda D_\lambda M_\alpha$ and therefore $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \nsubseteq \delta_{Ext}(M_\alpha)$. Hence, $\delta_{Ext}(M_\alpha) = \{1\}$, which completes proof (2).
For the composition operator defined by equation (2.3), given that $|\lambda| \geq 1$ and $D_{\frac{1}{\lambda}}$ is the diagonal operator, then

$$D_{\frac{1}{\lambda}}(x_1, x_2, x_3, ...) \;=\; (x_1, \frac{1}{\lambda}x_2, \frac{1}{\lambda^2}x_3, ...),$$

and so

$$\begin{aligned} C_\alpha D_{\frac{1}{\lambda}}(x_1, x_2, x_3, ...) &= C_\alpha(x_1, \frac{1}{\lambda}x_2, \frac{1}{\lambda^2}x_3, ...) \\ &= (0, \alpha_1 x_1, \frac{1}{\lambda}\alpha_2 x_2, \frac{1}{\lambda^2}\alpha_3 x_3, ...); \end{aligned}$$

while

$$\begin{aligned} \lambda D_{\frac{1}{\lambda}} C_\alpha(x_1, x_2, x_3, ...) &= \lambda D_{\frac{1}{\lambda}}(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, ...) \\ &= \lambda(0, \frac{1}{\lambda}\alpha_1 x_1, \frac{1}{\lambda^2}\alpha_2 x_2, \frac{1}{\lambda^3}\alpha_3 x_3, ...) \\ &= (0, \alpha_1 x_1, \frac{1}{\lambda}\alpha_2 x_2, \frac{1}{\lambda^2}\alpha_3 x_3, ...). \end{aligned}$$

Thus

$$C_\alpha D_{\frac{1}{\lambda}} \;=\; \lambda D_{\frac{1}{\lambda}} C_\alpha,$$

implying that $\{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \subseteq \delta_{Ext}(C_\alpha\})$.
Now suppose $|\lambda| < 1$ and D_λ is the diagonal operator where

$$D_\lambda(x_1, x_2, x_3, ...) \;=\; (x_1, \lambda x_2, \lambda^2 x_3, ...).$$

Then

$$\begin{aligned} C_\alpha D_\lambda(x_1, x_2, x_3, ...) &= C_\alpha(x_1, \lambda x_2, \lambda^2 x_3) \\ &= (0, \alpha_1 x_1, \lambda\alpha_2 x_2, \lambda^2\alpha_3 x_3, ...), \end{aligned}$$

and

$$\begin{aligned} \lambda D_\lambda C_\alpha(x_1, x_2, x_3, ...) &= \lambda D_\lambda(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, ...) \\ &= \lambda(0, \lambda\alpha_1 x_1, \lambda^2\alpha_2 x_2, \lambda^3\alpha_3 x_3, ...) \\ &= (0, \lambda^2\alpha_1 x_1, \lambda^3\alpha_2 x_2, \lambda^4\alpha_3 x_3, ...). \end{aligned}$$

Thus,

$$C_\alpha D_\lambda \;\neq\; \lambda D_\lambda C_\alpha.$$

implying that $\lambda \notin \delta_{Ext}(C_\alpha)$.
Therefore $\{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \subseteq \delta_{Ext}(C_\alpha)$ which completes the proof of (3). □

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