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NUMERICAL SOLUTION OF DYNAMIC VIBRATION EQUATIONS

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ABSTRACT

In this paper, we examine conservative autonomous dynamic vibration equation, $\ddot{x} = -tanh^2 x$, which is time vibration of the displacement of a structure due to the internal forces, with no damping or external forcing. Numerical results using Newmark are tabulated. The stability of the algorithm employed is also discussed.

Keywords: *Numerical solution, Dynamic vibration equation, stability*

1. INTRODUCTION

Most structures are in a continuous state of dynamic motion because of random loading such as wind, vibration equipment, or human loads. Therefore a lot of consideration has been given in the design of certain facilities or structures which need to resist sudden but strong vibrations. Small surrounding vibrations are normally near the natural frequencies of the structure and are terminated by energy dissipation in the real structure. In this study of dynamic system we are mainly interested in examining the time vibration of the displacement of a structure due to the internal forces, with on damping or external forcing. Practically, vibrations decay with time but in theory these vibrations do not actually decay. For vibrations due to purely internal forces, the dynamic systems are referred to as conservative systems. The methods of solution adopted for solving non-linear single-degree-of-freedom problems may be extended to multi-degree-of-freedom problems. There are many studies in literature on the application of these methods of solution to linear problems and yet so few have been applied to the non-linear problems.

2. NON-LINEAR CONSERVATIVE AUTONOMOUS SECOND ORDER SYSTEM

Let us consider the non-linear conservative autonomous second order system equation which is generally given by

$$\eta\ddot{x} = -\mu\dot{x} - \tau f(x) \tag{2.0.1}$$

with some initial conditions $x(0) = \alpha_0$ and $\dot{x} = \alpha_1$, where η, μ and τ are real positive numbers and $-\mu\dot{x}$ is the damping force. As an explanation to the new terms, we note that:

(a) The system is conservative because dynamic systems obey the principal of conservation of energy which asserts that the sum of kinetic and potential energies is constant in a conservative force of field.

(b) The system is autonomous because we are concerned with a system of ordinary differential equations which does not explicitly but implicitly contain the independent variable t (time).

(c) The restoring force, $f(x)$, defines the position of the moving object from its equilibrium point.

(d) There is no damping force i.e. no resisting medium so that $-\mu\dot{x} = 0$.

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Thus substituting $\eta = 1$, $-\mu\dot{x} = 0$ and $\tau = 1$ in (2.0.1) we have

$$\ddot{x} = -f(x) \tag{2.0.2}$$

and for this study let us consider

$$f(x) = tanh^2 x \tag{2.0.3}$$

the dynamic vibration equation

$$\ddot{x} = -f(x), x(0) = \alpha_1 \text{ and } \dot{x}(0) = \alpha_1 \text{ at } t = 0 \tag{2.0.4}$$

From equation (2.0.2), we can derive an autonomous system, in the form of

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = -f(x) \tag{2.0.5}$$

Where the right hand does not involve t explicitly but implicitly through the fact that x and y themselves depend on t and thus being self-governing. The above reduction of second order non-linear to equivalent first order non-linear is by introducing a new independent variable

$$y = \frac{dx}{dt}$$

and since $\frac{dy}{dt} = \frac{d^2x}{dt^2}$

the variables x and y satisfy the equivalent first-order system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -f(x)$$

Where equivalent means that such solution to the first order system uniquely corresponds to a solution to the second order equation and vice versa.

Specifically, equation (2.0.2) is equivalent to the autonomous system

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = -f(x), x(0) = \alpha_0 \text{ and } y = \dot{x}(0) = \alpha_1 \text{ at } t = t_0 \tag{2.0.6}$$

From (2.0.6)

$$\frac{dy}{dt} = -\frac{f(x)}{y} \tag{2.0.7}$$

$$\Rightarrow \int_{\alpha_1}^{y} y dy = -\int_{\alpha_0}^{x} f(x)dx \text{ or } \frac{y^2}{2} - \frac{\alpha_1^{\ 2}}{2} = -\{\int_{\alpha_0}^{0} f(x)dx + \int_0^x f(x)dx\}$$

$$\frac{y^2}{2} + \int_0^x f(x)dx = \frac{\alpha_1^{\ 2}}{2} + \int_0^{\alpha_0} f(x)dx \tag{2.0.8}$$

Thus

$$KE + PE = c$$

where $KE - \frac{y^2}{2}$ is the kinetic energy of the dynamic system (2.0.2), $PE = \int_0^x f(x)dx$ is the potential energy of the dynamic system (2.0.2) while $c = \frac{\alpha_1^{\ 2}}{2} + \int_0^{\alpha_0} f(x)dx$ is the constant (energy level). So equation (2.0.8) expresses the law of conservation of energy.

For the physical interpretation of the study, the nonlinear restoring force, $f(x)$ above, gives rise to special cases of nonlinear spring motion according to its behavior.

Equation (2.0.2) is said to represent
(i) A 'hard' spring if the magnitude of the restoring force, $f(x)$ acting on the mass, does increase more rapidly than that of a linear spring.
(ii) A 'soft' spring if the magnitude of the restoring force, $f(x)$ acting on the mass, does

increase less rapidly than that of a linear spring.

The above mentioned two special cases of equation (2.0.2) form the central subject of discussion in this paper.

Considering the function (2.0.2) and another situation where the restoring force is

$$f(x) = x^5 + x \tag{2.0.9}$$

We have two cases: (see table 1)

(a) $\frac{d^2x}{dt^2} + x^5 + x = 0$

(b) $\frac{d^2x}{dt^2} + tanh^2x = 0$

x	0	0.5	1	1.5	2	2.5	3
$x^5 + x$	0	0.5313	2	9.0938	34	100.1563	246
$tanh^2x$	0	0.2136	0.5800	0.8193	0.9298	0.9734	0.9901

Table 1: The table of restoring forces

The trends in the above table depict clearly the idea of the 'hard' spring and 'soft' spring for the two nonlinear restoring forces given.

Considering the magnitude of the nonlinear restoring force, $f(x) = x^5 + x$ in case (a), since it does increase more rapidly than that of a linear spring i.e. $f(x) = x$, it represents a 'hard' spring.

On the other hand, considering the magnitude of the nonlinear restoring force, $f(x) = tanh^2x$ in case (b), since it does increase less rapidly than that of a linear spring i.e. $f(x) = x$, it represents a 'soft' spring.

3. NEWMARK'S ALGORITHM

Consider the equation (2.0.2) given by $\ddot{x} + f(x) = 0$.Newmark's originally proposed method applied to.it is of the form

$$x_{n+1} = x_n + h\dot{x}_n + \left(\frac{1}{2} - \beta\right) h^2\ddot{x}_n + \beta h^2\ddot{x}_{n+1} \tag{3.0.10}$$

$$\dot{x}_{n+1} - \dot{x}_n + (1-\gamma) h\ddot{x}_n + \gamma h\ddot{x}_{n+1} \tag{3.0.11}$$

$$\ddot{x}_n + f(x_n) = 0 \text{ or } \ddot{x}_n + f_n = 0 \tag{3.0.12}$$

where h is the time step, and β and γ are the two Newmark parameters.

Write $n = 1$ for n into (3.0.12) to get

$$\ddot{x}_{n+1} + f_{n+1} = 0 \tag{3.0.13}$$

Eliminating $\ddot{x}_{n+1}$ from (3.0.12) and 3.0.10) we have

$$x_{n+1} + h^2\beta f_{n+1} = x_n + h\dot{x}_n + h^2(\tfrac{1}{2} - \beta)\ddot{x}_n \quad (3.0.14)$$

Similarly, when we eliminate $\ddot{x}_{n+1}$ from (3.0.12) and (3.0.11) we have

$$\dot{x}_{n+1} + \gamma h f_{n+1} = (1-\gamma)h\ddot{x}_n + \dot{x}_n \quad (3.0.15)$$

Eliminate $\dot{x}_n$ from (3.0.14) and (3.0.15) i.e. $(3.0.14) - h(3.0.15)$ we have

$$x_{n+1} + (\beta h^2 - \gamma h^2)f_{n+1} = h\dot{x}_{n+1} + h^2\left(\gamma - \beta - \tfrac{1}{2}\right)\ddot{x}_n + x_n \quad (3.0.16)$$

Write $n+1$ for n in (3.0.14)

$$x_{n+2} + cf_{n+2} = x_{n+1} + h\dot{x}_{n+1} + b\ddot{x}_{n+1} \quad (3.0.17)$$

Where $a = (1-\gamma)h, b = h^2(\frac{1}{2} - \beta)$ and $c = h^2\beta$.

Eliminate x_{n+1} from (3.0.17) and (3.0.15) and hence by using equation (3.0.14)

$$x_{n+2} + cf_{n+2} = 2x_{n+1} - (b + \gamma h^2)f_{n+1} + cf_{n+1} + (b - ah)f_n - x_n,$$

Or

$$x_{n+2} + h^2\beta f_{n+2} = 2x_{n+1} + \left(2\beta - \gamma - \tfrac{1}{2}\right)h^2 f_{n+1} - (x_n - (\gamma - \beta - \tfrac{1}{2})h^2 f_n \quad (3.0.18)$$

$x_{n+2} + h^2\beta f_{n+2} = Q$ where Q is known value

The scheme (3.0.18) in the displacement only is a two-step (three-time-level) scheme.

For $\beta = 0$ the scheme (3.0.18) becomes explicit, i.e.

$$x_{n+2} = 2x_{n+1} - \left(\gamma + \tfrac{1}{2}\right)h^2 f_{n+1} - \left(x_n - \left(\gamma - \tfrac{1}{2}\right)h^2 f_n\right) \quad (3.0.19)$$

The maximum accuracy for equation (2.0.2) is achieved when $\beta_0 = \frac{1}{6}$ i.e. $\beta = \frac{1}{12}$ and $\beta_1 = \frac{1}{2}$ i.e. $\gamma = \frac{1}{2}$. This is the Trapezium rule for the linear case.

Substitute $x_{n+1}, \dot{x}_{n+1}$ in our test equation (2.0.2) and we have

$$\ddot{x}_{n+1} + f\left(x_n + h\dot{x}_n + h^2\left(\tfrac{1}{2} - \beta\right)\ddot{x}_n + h^2\beta\ddot{x}_{n+1}\right) = 0$$

Or

$$\ddot{x}_{n+1} + f(a_0 + h^2\beta\ddot{x}_{n+1}) = 0 \quad (3.0.20)$$

where $a_0 = x_n + h\dot{x}_n + h^2(\frac{1}{2} - \beta)\ddot{x}_n$ is known value. Equation (2.0.20) is nonlinear in $\ddot{x}_{n+1}$ (or implicit) provided $\beta \neq 0$, and requires a nonlinear iterative method such as Newton-Raphson for solution.

4. NUMERICAL RESULTS OF THE EQUATION

Using the numerical algorithm developed above, equation (3.0.18) yields the required numerical results as follows: Considering the scheme (3.0.18) i.e. $x_{n+2} + h^2\beta tanh^2 x_{n+2} = Q$ where $Q = 2x_{n+1} + \left(2\beta - \gamma - \frac{1}{2}\right)h^2 tanh^2 x_{n+1} - (x_n - (\gamma - \beta - \frac{1}{2})h^2 tanh^2 x_n$) which is the displacement only and two-step (three-time-level) scheme. Using C++ computer programming with Newton-Raphson's iteration,

$x_{n+2} = x_{n+1} - \frac{F(x_{n+1})}{F'(x_{n+1})}$ from scheme (3.0.18)

$F(x_{n+1}) = x_{n+1} + h^2\beta tanh^2(x_{n+1}) - Q$, thus $F'(x_{n+1}) = 1 + 2h^2\beta \tanh(x_{n+1})\, sech^2(x_{n+1})$

leading to $x_{n+2} = x_{n+1} - \frac{x_{n+1} + h^2\beta tanh^2(x_{n+1}) - Q}{1+2h^2\beta \tanh(x_{n+1}) sech^2(x_{n+1})}$ where $sech^2((x)$ is expressed as $\frac{1}{cosh^2(x)}$ to be used in the computer programme.

The following conditions were taken into account when compiling results: it is clearly stated just that before equation (3.0.20) that the maximum accuracy is achieved when $\beta = \frac{1}{12}$ and $\gamma = \frac{1}{2}$ and so our choice of the parameters was influenced by the given parameters. Secondly, the stability of the numerical schemes is governed by small step size, h. Given that $x_1 = x_0 + h\dot{x}_0$, let $x_0 = 0.2$ and $\dot{x}_0 = 0.15$, thus $x_1 = 05.2 + 0.1(1.5) = 0.3$ leading to the following results:

t	x when β = 0.2 and γ = 0.1	x when $\beta = \frac{1}{12}$ and $\gamma = \frac{1}{2}$	x when β = 0.05 and γ = 0.4
0	0.2	0.2	0.2
0.1	0.35	0.35	0.35
0.2	0.478593	0.471592	0.473478
0.3	0.567135	0.54544	0.550679
0.4	0.598426	0.558432	0.567087
0.5	0.562833	0.508191	0.518565
0.6	0.461835	0.403806	0.412827
0.7	0.308038	0.262878	0.267165
0.8	0.121116	0.104905	0.102021
0.9	-0.0802323	-0.0567865	-0.067801
1.0	-0.288008	-0.220356	-0.239619
1.1	-0.511782	-0.396754	-0.424675
1.2	-0.779739	-0.609934	-0.648008
1.3	-1.13535	-0.898417	-0.949256
1.4	-1.63076	-1.31534	-1.38177
1.5	-2.31625	-1.91814	-2.00176
1.6	-3.22819	-2.74816	-2.84913
1.7	-4.38406	-3.82302	-3.94118
1.8	-5.78896	-5.14714	-5.28242
1.9	-7.44377	-6.72119	-6.87359
2.0	-9.34856	-8.54524	-8.71475
2.1	-11.5034	-10.6193	-10.8059
2.2	-13.9082	-12.9433	-13.1471
2.3	-16.563	-15.5174	-15.7382
2.4	-19.4678	-18.3414	-18.5794
2.5	-22.6226	-21.4155	-21.6706
2.6	-26.0273	-24.7395	-25.0117
2.7	-29.6821	-28.3136	-28.6029
2.8	-33.5869	-32.1376	-32.4441
2.9	-37.7417	-36.2117	-36.5352
3.0	-42.1465	-40.5357	-40.8764

5. STABILITY OF THE NUMERICAL ALGORITHM

From the result of the two-step (three-time-level) scheme tabulated above, it is clearly evident that for zero damping or no damping dynamic equation, the Newmark method is conditionally stable when the parameter chosen, for instance $\beta = 0.05$ and $\gamma = 0.4$ are within the neighbourhood of the parameters associated with maximum accuracy i.e. $\beta = \frac{1}{12}$ and $\gamma = \frac{1}{2}$. As we move away from maximum accuracy parameters, the method no longer conditionally stable.

6. CONCLUSION

In this paper, we have looked at implicit nonlinear numerical scheme that can be used to solve the implicit nonlinear dynamic vibration equations. In the nonlinear dynamic method applied, that is, Newmark method, we used a displacement only, two-step (three-time-level) scheme with a C++ computer programming which is fast and accurate in producing results. The equation has been solved using Newton-Raphson iteration method which converges fast to a meaningful solution. The results are tabulated. For the stability of the numerical schemes, a small step size is needed, with maximum accuracy achieved when the Newmark parameters, β and γ are $\frac{1}{12}$ and $\frac{1}{2}$ respectively. The results of our study indicate that Newmark algorithm exhibit stable case for the solution of the softening spring, equation (2.0.4) when parameters chosen are very close to the maximum accuracy parameters, otherwise unstable when parameters chosen are not close to the maximum accuracy parameter.

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