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On certain norm inequalities for inner product type integral transformers

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Abstract: Studies on inner-product-type integral transformers have been considered in many research works from various perspectives, including spectra, numerical ranges, and operator inequalities. An open problem remains concerning inequalities related to norm estimates for inner-product-type integral transformers whose spectra are contained in the unit disc. It has been observed that the norms of such transformers can be attained under the condition that one of the implementing operators is normal. In this note, we address this problem by establishing norm inequalities for inner-product-type integral transformers in a general Banach space setting.

Keywords: inner product type elementary operator, norm, inequality

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1. Introduction

Several studies on inner product type integral transformers have been carried out from various perspectives, including geometry, spectra [1], numerical ranges, and operator inequalities. The work in [2] considered Cauchy–Schwarz inequalities for operators defined as weak integrals. In that work, finite measures played a central role in the study of measurable families, and Cauchy–Schwarz inequalities were described for non-commutative operators in Schatten P-ideals.

Unitarily invariant norms for these weak integrals were also established by applying the arithmetic mean inequality and Young's inequality in [3]. That study also presented several definitions related to operator-valued functions and measurable functions. It further included applications involving the closed graph theorem, as discussed in [4].

The study also employed concepts related to vectors that are integrable with respect to operator-valued functions, as discussed in [5]. Gelfand axioms also played an important role in that work and enabled a full description of weak integrals. The results provided detailed information, particularly on the norms of operators in Banach spaces, by using fundamental Cauchy–Schwarz inequalities, as illustrated in [6].

In [7], advances and reverse inequalities for the well-known Cauchy–Schwarz inequality were discussed. These reverses were centered on Cauchy–Schwarz inequalities for inner product spaces and C^*-modules, and a comprehensive description was given for the role of Cauchy–Schwarz inequalities in classical analysis.

For instance, some proofs for inner product spaces were presented, especially those arising from the parallelogram law [8]. The authors used a principal argument for non-zero vectors obtained through the normalization process in order to derive results for the classical analysis case. They also employed Wagner's inequality for the operator version, self-adjoint operators, and C^*-algebras.

The study in [9] described Cauchy–Schwarz inequalities for general elementary operators and also discussed their applications. The authors employed unitarily invariant norms for positive real scalars. In doing so, they established a strong relationship between known versions of the Cauchy–Schwarz inequalities and used this relationship to calculate the corresponding norms.

In the same work, it was shown that the symmetric gauge function is a basic property associated with the singular values of operators. The absolute singular values were also obtained. The study further presented several theorems describing Cauchy–Schwarz inequalities for positive semidefinite operators, as observed in [10].

Regarding various types of norms, the study in [11] considered unitarily invariant norms involving normal operators. In these studies, upper norm bounds for operators were represented by using unitarily invariant norms.

The study in [12] discussed norms for inner product type integral transformers whose spectra lie in the unit disc. In that study, the authors introduced a weakly-$*$ measurable family and used $B(H)$ to denote the algebra of all bounded linear operators on a Hilbert space H.

In [13], the authors extensively discussed norm estimates of operators in relation to contractions and spectra. The study postulated that, given any two operators A and B as normal contractions on Hilbert spaces, their norms can be obtained in terms of partial summations involving column matrices.

In [7], the author worked extensively on applications of Cauchy–Schwarz inequalities for Hilbert modules to inner product type integral transformers. The work in [8] later considered norm interpolations for row and column elementary operators and described the formula for the Hilbert–Schmidt norm of inner product type integral transformers.

Recently any two operators A and B as normal contractions on Hilbert spaces, their norms can be obtained in terms of partial summations involving column matrices.

In [7], the author worked extensively on applications of Cauchy–, other studies, such as [9], discussed Landau and Gr''uss inequalities for inner product type integral transformers in general Hilbert spaces. That study gave a description of norm inequalities with respect to unitarily invariant norms in compact separable Hilbert spaces.

The study also dealt with operator-valued functions over a measure space $(\Omega, \mathcal{M}, \mu)$ for a finite measure μ. It further presented a generalization of Landau's inequality for Gelfand integrals of operator-valued functions in unitarily invariant norm ideals.

The study in [4] continued to explore norm estimates for positive normal contractions. The work utilized the defect operator $\triangle_{A\alpha}$ and obtained similar inequalities in the same space in a more compact form. However, that study did not consider operators on infinite-dimensional Hilbert spaces.

The author also focused on the Q-norm of commuting normal operators, with some applications of norms for $P \geq 2$ to the partial normality of operators acting on Banach spaces. It is clear that these studies do not fully consider the norm property for the inner product type integral transformers addressed in this note.

Throughout this paper, we use the following notation: IPTIT denotes an inner product type integral transformer; $B(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H; $B(H_n)$ denotes the algebra of all bounded linear operators on an n-tuple of Hilbert spaces; $B(H_1, H_2)$ denotes the algebra of all bounded linear operators from the Hilbert space H_1 to the Hilbert space H_2; $B_{\text{IPTIT}}(B(H_1, H_2))$ denotes the algebra of all IPTITs on $B(H_1, H_2)$; $|\cdot|_{tr}$ denotes the tracial norm; and tr denotes the trace.

2. Preliminaries

This section gives a description of some definitions which are useful to this work.

Definition 1 ([7])**.** Let H be a complex Hilbert space with infinite dimension, then for $B(H)$ representing all linear and bounded operators on H, we have a mapping S represented as

$$S_{A_i,B_i}(X) = \sum_{i=1}^{k} A_i X B_i,$$

called the elementary operator, where x is arbitrary in $B(H)$ while a_i, b_i are fixed in $B(H)$.

Definition 2 ([6])**.** Consider a weakly μ-measurable operator valued functions $M, N : \Omega \to B(H)$. For any $Q \in B(H)$, we can have a mapping $Q \mapsto \int_\Omega M_t Q N_t$ as weakly measurable too. If M and N are integrable with respect to Gelfand axiom, then the linear transformation arising from the inner product space as given by $Q \mapsto \int_\Omega M_t Q N_t \, d(t)$ is called an inner product type integral transformer denoted by $\int_\Omega M_t \otimes N_t \, d(t)$.

3. Main results

We carry out a detailed analysis and obtain new norm inequalities for inner product type integral transformers. We begin with the following auxiliary proposition.

Proposition 1. *Let H be a complex Hilbert space, and let $B(H_n)$ and $B(H_1, H_2)$ be as defined above. Suppose that $X \in B(H_n)$, and let $S_1, S_2 \in B(H_1, H_2)$ be weakly-$*$ measurable.*

(a) *If $f \in H$ and*

$$\int_\Omega \|S_{1t}^* f\|^2 \, d\mu(t) < \infty,$$

then the following hold:

(i) *The mapping $t \mapsto S_{1t} X S_{2t}$ is weakly-$*$ integrable, and*

$$\left\| \int_\Omega S_1 X S_2 \, d\mu \right\| \le \sqrt{\left\| \int_\Omega S_1 S_1^* \, d\mu \right\| \left\| \int_\Omega S_2^* S_2 \, d\mu \right\|} \|X\|.$$

(ii) *If $\{S_{1n}^*\}_{n=1}^\infty$ and $\{S_{2n}\}_{n=1}^\infty$ are sequences of simple operator-valued functions for S_1^* and S_2, respectively, then*

$$\int_\Omega S_{1n} X S_{2n} \, d\mu \to \int_\Omega S_1 X S_2 \, d\mu$$

weakly as $n \to \infty$.

(b) *If*

$$\int_\Omega \left(\|S_{1t} f\|^2 + \|S_{2t}^* f\|^2 \right) d\mu(t) < \infty$$

for all $f \in H$, then

$$\left\| \int_\Omega S_1 X S_2 \, d\mu \right\| \le \sqrt{\left\| \int_\Omega S_1^* S_1 \, d\mu \right\| \left\| \int_\Omega S_2 S_2^* \, d\mu \right\|} \|X\|, \quad \forall X \in B(H_1, H_2).$$

(c) *If*

$$\int_\Omega \left(\|S_{1t} f\|^2 + \|S_{1t}^* f\|^2 + \|S_{2t} f\|^2 + \|S_{2t}^* f\|^2 \right) d\mu(t) < \infty$$

for all $f \in H$, then

$$\left\| \int_\Omega S_1 X S_2 \, d\mu \right\| \le \sup \left\{ \sqrt{\left\| \int_\Omega S_1^* S_1 \, d\mu \right\| \left\| \int_\Omega S_2 S_2^* \, d\mu \right\|} \sqrt{\left\| \int_\Omega S_1 S_1^* \, d\mu \right\| \left\| \int_\Omega S_2^* S_2 \, d\mu \right\|} \|X\| \right\},$$

for all $X \in B(H)$.

Proof. *Case (a)(i):* Clearly, if

$$|\langle S_{1t} X S_{2t} f, f \rangle| \le \|X\| \, \|S_{1t}^* f\| \, \|S_{2t} f\|$$

for all $t \in \Omega$ and $f \in H$, then the mapping $t \mapsto \langle S_{1t} X S_{2t} f, f \rangle$ is integrable. Hence, $S_{1t} X S_{2t}$ is weakly-$*$ integrable. Therefore,

$$\begin{aligned}
\left| \int_\Omega \langle S_1 X S_2 f, g \rangle \, d\mu \right| &\le \|X\| \int_\Omega \|S_{1t}^* g\| \, \|S_{2t} f\| \, d\mu(t) \\
&\le \|X\| \sqrt{\int_\Omega \|S_{1t}^* g\|^2 \, d\mu(t)} \sqrt{\int_\Omega \|S_{2t} f\|^2 \, d\mu(t)} \\
&= \|X\| \sqrt{\left\langle \left(\int_\Omega S_1 S_1^* \, d\mu \right) g, g \right\rangle} \sqrt{\left\langle \left(\int_\Omega S_2^* S_2 \, d\mu \right) f, f \right\rangle} \\
&\le \sqrt{\left\| \int_\Omega S_1 S_1^* \, d\mu \right\| \left\| \int_\Omega S_2^* S_2 \, d\mu \right\|} \|X\| \, \|f\| \, \|g\|.
\end{aligned}$$

Case (a)(ii): For all $f, g \in H$, we have

$$\begin{aligned}
&\left|\int_\Omega \langle (S_{1m}XS_{2m} - S_1XS_2)f, g\rangle \, d\mu\right| \\
&\leq \|X\| \int_\Omega \Big(\|S_{1m}^*g - S_1^*g\| \, \|S_{2m}f\| + \|S_1^*g\| \, \|S_{2m}f - S_2f\| \Big) \, d\mu(t) \\
&\leq \|X\| \sqrt{\int_\Omega (\|S_1^*g\|^2 + \|S_{2m}f\|^2) \, d\mu(t)} \sqrt{\int_\Omega (\|S_{1m}^*g - S_1^*g\|^2 + \|S_{2m}f - S_2f\|^2) \, d\mu(t)} \to 0.
\end{aligned}$$

Thus,

$$\int_\Omega S_{1n}XS_{2n} \, d\mu \to \int_\Omega S_1XS_2 \, d\mu$$

weakly as $n \to \infty$.

Case (b): The proof of this part follows directly from the duality of norms.

Case (c): We have

$$T_{S_1,S_2}(X) = \int_\Omega S_1XS_2 \, d\mu \in \mathfrak{B}_{\text{IPTIT}}(B(H_1, H_2))$$

whenever X is a finite-rank operator. This agrees with the statement of the proposition, namely,

$$\left\|\int_\Omega S_1XS_2 \, d\mu\right\| \leq \sqrt{\left\|\int_\Omega S_1^*S_1 \, d\mu\right\| \left\|\int_\Omega \Xi \, d\mu\right\|} \sqrt{\left\|\int_\Omega \Pi \, d\mu\right\| \left\|\int_\Omega S_2^*S_2 \, d\mu\right\|} \|X\|, \quad \forall X \in H,$$

where we set $\Pi = S_1S_1^*$ and $\Xi = S_2S_2^*$ for simplicity. □

In the next proposition, we consider a norm inequality involving commuting families of measurable normal operators, as stated below.

Proposition 2. *Let $S_1, S_2 \in B(H_1, H_2)$ be such that $\{S_{1t}\}_{t\in\Omega}$ and $\{S_{2t}\}_{t\in\Omega}$ are commuting families of measurable normal operators. For all $f \in H$, assume that*

$$\int_\Omega (\|S_{1t}f\|^2 + \|S_{2t}f\|^2) \, d\mu(t) < \infty.$$

Moreover, for $\widehat{T}_{S_1,S_2} \in \mathfrak{B}_{\text{IPTIT}}(B(H_1), B(H_2))$, we have

$$\left\|\widehat{T}_{S_1,S_2}(X)\right\| \leq \left\|\sqrt{\int_\Omega S_1^*S_1 \, d\mu}\right\| \left\|\sqrt{\int_\Omega S_2^*S_2 \, d\mu}\right\|.$$

Proof. Since $\{S_{1m}\}_{m=1}^\infty$ and $\{S_{2m}\}_{m=1}^\infty$ are sequences, we have

$$\begin{aligned}
\left\|\widehat{T}_{S_1,S_2}(X)\right\| &= \left\|\int_\Omega S_{1m}XS_{2m} \, d\mu\right\| \\
&\leq \left\|\sqrt{\int_\Omega |S_{1m}|^2 \, d\mu} \times \sqrt{\int_\Omega |S_{2m}|^2 \, d\mu}\right\| \\
&\leq \left\|\sqrt{\int_\Omega |S_1|^2 \, d\mu} \times \sqrt{\int_\Omega |S_2|^2 \, d\mu}\right\|.
\end{aligned}$$

By the monotonicity of norms and Proposition 1, lower semicontinuity is sufficient. Without loss of generality, we have

$$\left\|\sqrt{\int_\Omega |S_1|^2 \, d\mu} \times \sqrt{\int_\Omega |S_2|^2 \, d\mu}\right\| = \left\|\sqrt{\int_\Omega S_1^*S_1 \, d\mu} \times \sqrt{\int_\Omega S_2^*S_2 \, d\mu}\right\|.$$

Hence,

$$\left\|\widehat{T}_{S_1,S_2}(X)\right\| \leq \left\|\sqrt{\int_\Omega S_1^*S_1 \, d\mu} \times \sqrt{\int_\Omega S_2^*S_2 \, d\mu}\right\|.$$

□

In the next lemma, we consider a norm inequality involving commuting families of measurable normal operators in the Hilbert space setting.

Lemma 1. *Let $X \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$, and let $\{S_{1t}\}_{t\in\Omega}$ and $\{S_{2t}\}_{t\in\Omega}$ be separable families of measurable normal operators such that*

$$\int_\Omega S_1^* S_1 \, d\mu \le 1 \quad \text{and} \quad \int_\Omega S_2^* S_2 \, d\mu \le 1.$$

Then

$$\left\| \sqrt{1 - \int_\Omega S_1^* S_1 \, d\mu} \times \sqrt{1 - \int_\Omega S_2^* S_2 \, d\mu} \right\| \le \left\| X - \int_\Omega S_1 X S_2 \, d\mu \right\|,$$

for all $X \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$.

Proof. For normed ideals $I \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$, we have

$$X - I_{S_1,S_2} X \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2)).$$

Therefore, for all $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\left\| \sqrt{1 - \int_\Omega S_1^* S_1 \, d\mu} \sqrt{1 - \int_\Omega S_2^* S_2 \, d\mu} \right\| \\
&\le \left\| \sqrt{I - S_1^2} \sqrt{I - S_2^2} \right\| \\
&\le \left\| \sqrt{I - S_1^2} (X - S_{S_1,S_2} X) \sqrt{I - S_2^2} \right\| + \left\| \sqrt{I - S_1^2} I^n_{S_1,S_2} \sqrt{I - S_2^2} \right\| \\
&= \sum_{i=1}^{k} I^l_{S_1,S_2} \left\| \sqrt{I - S_1^2} (X - S_{S_1,S_2} X) \sqrt{I - S_2^2} \right\| + \left\| I_{S_1,S_2} \sqrt{I - S_1^2} \sqrt{I - S_2^2} \right\| \\
&= \sum_{i=1}^{k} I^{l^2}_{S_1} \left\| \sqrt{I - S_1^2} (X - S_{S_1,S_2} X) \sqrt{I - S_2^2} \right\| \sqrt{\sum_{i=1}^{k} I^{l^2}_{S_1}} + \left\| S_1^m \sqrt{I - S_1^2} \sqrt{I - S_2^2} \right\| \\
&\le \left\| \sqrt{I - S_1^{2m}} (X - S_{S_1,S_2} X) \sqrt{I - S_2^m} \right\| + \left\| S_1 \sqrt{S_1^{2m} - S_1^{2m+2}} \right\| \left\| \sqrt{S_2^{2m} - S_2^2} \right\| \|X\| \\
&\le \|X - I_{S_1,S_2}\| + \frac{1}{k+1} \|X\| \left\| X - \int_\Omega S_1 X S_2 \, d\mu \right\| + \frac{1}{k+1} \|X\|.
\end{aligned}$$

Since $\{S_{1t}\}_{t\in\Omega}$ consists of normal operators, the commutativity axiom is satisfied. Therefore,

$$\sqrt{\sum_{i=0}^{m-1} \int_{\Omega}, \int_{\Omega}, \ldots, \int_\Omega |S_{2t_1}, S_{2t_2}, \ldots, S_{2t_m}|^2 \, d\mu(t_1)} = \sqrt{\sum_{i=0}^{m-1} S_2^{2t}}.$$

Consequently, we have

$$\begin{aligned}
\left\| s_1^{2n} - S_2^{n+2} \right\| &\le \bigwedge_{t\in[0,1]} \left| t^{2n} - t^{2n+2} \right| \\
&= \frac{k}{(k+1)^{k+1}} \\
&= \frac{1}{k+1}.
\end{aligned}$$

Since n is arbitrary, it follows from the statement of the lemma that

$$\left\| \sqrt{1 - \int_\Omega S_1^* S_1 \, d\mu} X \sqrt{1 - \int_\Omega S_2^* S_2 \, d\mu} \right\| \le \left\| X - \int_\Omega S_1 X S_2 \, d\mu \right\|,$$

for all $X \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$. $\square$

This leads us to the next theorem, which involves inequalities for separable families of measurable normal operators with respect to an orthonormal basis.

Theorem 1. *Let $S_1, S_2 \in B(H_1, H_2)$ and $\widehat{T} \in \mathfrak{B}_{\text{IPTIT}}(B(H_1), B(H_2))$ be such that*

$$\int_\Omega \|S_{1t}\|_1 \|S_{2t}\|_1 \, d\mu(t) < \infty.$$

Then

$$\|\widehat{T}_{S_1,S_2}\|_{\text{tr}} = \int_\Omega \text{tr}(S_{1t}) \, \text{tr}(S_{2t}) \, d\mu(t).$$

Proof. Let $\{X_n\}_{n=1}^\infty \in B(H_1, H_2)$. Then $\{X_m \otimes X_n\}_{m,n=1}^\infty$ forms an orthonormal basis in $B(H_1, H_2)$. Therefore,

$$\begin{aligned}
\|\widehat{T}_{S_1,S_2}\|_{\text{tr}} &= \sum_{m,n=1}^\infty \left\langle \int_\Omega S_{1t}(x_m \otimes x_n) S_{2t} \, d\mu(t), x_m^* \otimes x_n \right\rangle_{B(H_1,H_2)} \\
&= \sum_{m,n=1}^\infty \text{tr} \int_\Omega (S_{2t}^* x_n)^* \otimes S_{1t} x_n \, d\mu(t) (x_n^* \otimes x_m) \\
&= \sum_{m,n=1}^\infty \text{tr} \int_\Omega \langle x_m, S_{2t} x_m \rangle x_n^* \otimes S_{1t} x_n \, d\mu(t) \\
&= \int_\Omega \sum_{m,n=1}^\infty \langle S_{1t} x_n, x_n \rangle \langle S_{2t} x_m, x_m \rangle \, d\mu(t) \\
&= \int_\Omega \text{tr}(S_{1t}) \, \text{tr}(S_{2t}) \, d\mu(t).
\end{aligned}$$

□

The next consequence follows immediately.

Corollary 1. *For all $S \in B(H_1, H_2)$, $\widehat{T} \in \mathfrak{B}_{\text{IPTIT}}(B(H_1), B(H_2))$, and $q \in H$, the inequality*

$$\|\widehat{T}_{S,S^*} X\|_q \leq \|\widehat{T}_{S,S^*}\|^{\frac{1}{q-1}} \|X\|_q, \quad \forall X \in B(H_1, H_2),$$

holds.

Proof. Recall that $(\widehat{T}_{S,S^*} X)^{q-1}$ and $\widehat{T}_{S,S^*}(\widehat{T}_{S^*,S} X)^{q-1}$ belong to $\mathfrak{B}_{\text{IPTIT}}(B(H_1), B(H_2))$. Hence, $\widehat{T}_{S,S^*}$ is bounded. Therefore,

$$\begin{aligned}
\|\widehat{T}_{S,S^*} X\|_q &= \|\widehat{T}_{S,S^*}(\widehat{T}_{S^*,S} X)\|_q \\
&= \|\widehat{T}_{S,S^*}(\widehat{T}_{S^*,S} X)^{q-1}\|_q \sqrt[n]{q} \\
&\leq \|\widehat{T}_{S,S^*}\|^{\frac{1}{q-1}} \|(\widehat{T}_{S,S^*} X)^{q-1}\|^{\frac{1}{q-1}} \\
&= \|\widehat{T}_{S,S^*}\|^{\frac{1}{q-1}} \|\widehat{T}_{S^*,S} X\|_q \\
&\leq \|\widehat{T}_{S,S^*}\| \|X\|_q.
\end{aligned}$$

□

The next result considers sequences of operators under a scalar multiple of a positive integer.

Proposition 3. *Let $\widehat{T}_{S_1,S_2} \in \mathfrak{B}_{\text{IPTIT}}(B(H_1, H_2))$, and suppose that, for all $q \geq 1$,*

$$\int_\Omega \|S_{1t}\|_q \|S_{2t}\|_q \, d\mu(t) < \infty.$$

Also, for all $S_1, S_2 \in B(H_1, H_2)$ and any integer $M \geq \frac{q}{2}$, we have

$$\left\| \widehat{T}_{S_1,S_2} \right\|_2^2 = \int_\Omega 2m \ \mathrm{tr} \left(\prod_{l=1}^{m} S^*_{1t_{m+1-l}} S_{1t_{m+1-k}} \right) \mathrm{tr} \left(\prod_{l=1}^{m} S_{2t_l}, S^*_{1t_l} \right) \prod_{l=1}^{m} d\mu(t_l), \tag{1}$$

and if

$$\left\| \widehat{T}_{S_1,S_2} \right\|_\infty = \lim_{m\to\infty} \sqrt[2m]{\int_\Omega 2m \ \mathrm{tr} \left(\prod_{l=1}^{m} S^*_{1t_{m+1-l}} S_{1t_{m+1-k}} \right) \mathrm{tr} \left(\prod_{l=1}^{m} S_{2t_l}, S^*_{1t_l} \right) \prod_{l=1}^{m} d\mu(t_l)},$$

then

$$\left\| \widehat{T}_{S_1,S_2} \right\|_{2m} \leq \int_\Omega \|S_{1t}\|_q \|S_{2t}\|_q \, d\mu(t). \tag{2}$$

Proof. Clearly,

$$\mathrm{tr}\, |\widehat{T}_{S_1,S_2}|^{2m} = \mathrm{tr}(\widehat{T}^*_{S_1,S_2}, \widehat{T}_{S_1,S_2})^n.$$

By Theorem 1, we observe that $(\widehat{T}^*_{S_1,S_2}, \widehat{T}_{S_1,S_2})^n$ is an inner product type integral transformer. Hence,

$$\left(\widehat{T}^*_{S_1,S_2}, \widehat{T}_{S_1,S_2}\right)^n X = \int_\Omega 2m \prod_{l=1}^{m} S^*_{1t_{m+1-l}} S_{1t_{m+1-k}} X \prod_{l=1}^{m} S_{2t_l}, S^*_{1t_l} \prod_{l=1}^{m} d\mu(t_l),$$

and from the norm equality (1) and inequality (2), we obtain

$$\begin{aligned}
\left\| \widehat{T}_{S_1,S_2} \right\|_{2m} &\leq \int_\Omega 2m \left\| \prod_{l=1}^{m} S^*_{1t_{m+1-l}} S_{1t_{m+1-k}} \right\| \left\| \prod_{l=1}^{m} S_{2t_l}, S^*_{1t_l} \right\| \prod_{l=1}^{m} d\mu(t_l) d\mu(t_k) \\
&\leq \int_\Omega 2m \prod_{l=1}^{m} \|S_{1t_l}\|_{2m} \|S_k\|_{2m} \prod_{l=1}^{m} \|S_{2k}\|_{2m} \|S^*_{2t_l}\|_{2m} \prod_{l=1}^{m} d\mu(t_l) d\mu(t_k) \\
&= \int_\Omega \|S_{1t}\|_{2m} \|S_{2t}\|_{2m} \, d\mu(t).
\end{aligned}$$

Hence, for any norm of the transformer $\widehat{T}_{S_1,S_2} \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$, we have

$$\widehat{T}_{S_1,S_2} \leq \int_\Omega \|S_{1t}\|_q \|S_{2t}\|_q \, d\mu(t) < \infty.$$

□

The next result considers inequalities for weakly-∗ measurable families.

Proposition 4. *Let $S_1, S_2 : \Omega \to B(H_1, H_2)$ be weakly-∗ measurable operator families. Then*

$$\bigvee_{n\in\mathbb{N}} \left\| \int_{\Omega^n} \|S_{1t_1}, \ldots, S_{1t_n}\|^2 \, d\mu^n(t_1, \ldots, t_n) \right\|^{\frac{1}{2}} \leq 1,$$

and

$$\inf_{n\in\mathbb{N}} \left\| \int_{\Omega^n} \|S_{2t_1}, \ldots, S_{2t_n}\|^2 \, d\mu^n(t_1, \ldots, t_n) \right\|^{\frac{1}{n}} \leq 1.$$

Also, let $\widehat{T} \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$, let $|||\cdot|||$ be a unitarily invariant norm, and assume that $X^*X \in B(H_1, H_2)$ for all $X \in B(H_1, H_2)$. If $\{S_{1t}\}_{t\in\mathbb{N}}$ is a sequence of normal operators, then the unitarily invariant norm satisfies

$$\left|\left|\left| \widehat{T}_{S_1} X S_2 \right|\right|\right|_2 = \left|\left|\left| X - \int_\Omega S_1^* X S_2 \, d\mu \right|\right|\right|_2.$$

Proof. Let $0 \le k < 1$. Then we obtain the expansion

$$\begin{aligned}
\left|\left|\left|\widehat{T}_{kS_1} X \widehat{T}_{kS_2}\right|\right|\right|_2 &= \left|\left|\left|\widehat{T}_{kS_1}\left(\sum_{n=0}^{\infty} k^{2n}\left(\int_\Omega S_1 \otimes S_2\, d\mu\right)^n \left(X - k^2 \int_\Omega S_1^* X S_2\, d\mu\right)\right)\widehat{T}_{kS_2}\right|\right|\right|_2 \\
&= \left|\left|\left|\sum_{n=0}^{\infty} k^{2n} \widehat{T}_{kS_1}^{(1-\frac{1}{q})} \int_{\Omega^n} S_1^{|n\rangle *}\left(X - k^2 \int_\Omega S_1^* X S_2^{|n\rangle}\, d\mu^n \widehat{T}_{kS_2}^{(1-\frac{1}{q})}\right)\right|\right|\right|_2 \\
&= \left|\left|\left|\sqrt{\widehat{T}_{kS_1} \sum_{n=0}^{\infty} k^{2n} \int_{\Omega^n} S_1^{|n\rangle *} S_2^{|n\rangle}\, d\mu^n \widehat{T}_{kS_1}\left(X - k^2 \int_\Omega S_1^* X S_2^{|n\rangle}\, d\mu\right)}\right|\right|\right|_2 \\
&\quad \left|\left|\left|\sqrt{\widehat{T}_{kS_2} \sum_{n=0}^{\infty} k^{2n} \int_{\Omega^n} S_2^{|n\rangle *} S_2^{|n\rangle}\, d\mu^n \widehat{T}_{kS_2}}\right|\right|\right|_2 \\
&= \left|\left|\left|X - k^2 \int_\Omega S_1^* X S_2\, d\mu\right|\right|\right|_2 .
\end{aligned}$$

Since this is a unitarily invariant norm, as assumed earlier, we obtain the lower semicontinuous norm as follows:

$$\begin{aligned}
\left|\left|\left|\widehat{T}_{S_1} X \widehat{T}_{S_2}\right|\right|\right|_2 &= \left|\left|\left|\omega - \lim_{k\uparrow 1} \widehat{T}_{kS_1} X \widehat{T}_{kS_2}\right|\right|\right|_2 \\
&= \lim_{k\uparrow 1} \left|\left|\left|\widehat{T}_{kS_1} X \widehat{T}_{kS_2}\right|\right|\right|_2 \\
&\le \lim_{k\uparrow 1} \left|\left|\left|X - k^2 \int S_1^* X S_2^*\, d\mu\right|\right|\right|_2 \\
&= \left|\left|\left|X - \int S_1^* X S_2^*\, d\mu\right|\right|\right|_2 .
\end{aligned}$$

□

Lemma 2. *Let $S_1, S_2 : \Omega \to B(H_1, H_2)$ be weakly-$*$ measurable operator-valued functions, and let $\widehat{T}_{S_1,S_2} \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$. Then, for all norm ideals in $B(H_1, H_2)$, we have*

$$\left\|\widehat{T}_{S_1,S_2}\right\|_{B(H_1)\to B(H_2)} \le \left|\left|\left|\int_\Omega S_{1t} S_{1t}^*\, d\mu(t)\right|\right|\right|_\Phi^{\frac{1}{2}} \left|\left|\left|\int_\Omega S_2^* S_2\, d\mu(t)\right|\right|\right|, \tag{3}$$

and

$$\left\|\widehat{T}_{S_1,S_2}\right\|_{B(H_2)\to B(H_1)} \le \left|\left|\left|\int_\Omega S_{1t}^* S_{1t}\, d\mu(t)\right|\right|\right|_\Phi^{\frac{1}{2}} \left|\left|\left|\int_\Omega S_{2t}^* S_{2t}\, d\mu(t)\right|\right|\right|, \tag{4}$$

where $|||\cdot|||_\Phi$ is the norm in the dual space of the compact operators in $B(H_1, H_2)$. In the special case $S_{2t} = S_{1t}^*$, inequalities (3) and (4) become

$$\left\|\widehat{T}_{S_1,S_2}\right\|_{B(H_1)\to B(H_2)} = \left|\left|\left|\widehat{T}_{S_1,S_2}(I)\right|\right|\right| = \left|\left|\left|\int_\Omega S_{1t} S_{1t}^*\, d\mu(t)\right|\right|\right|_\Phi, \tag{5}$$

and

$$\left\|\widehat{T}_{S_2,S_2^*}\right\|_{B(H_2)\to B(H_1)} = \left|\left|\left|\int_\Omega S_{1t}^* S_{1t}\, d\mu(t)\right|\right|\right|_\Phi, \tag{6}$$

respectively.

Proof. Since S_1 and S_2 are weakly-$*$ measurable operator-valued functions, for all $X \in B(H_1, H_2)$, the special case becomes

$$\begin{aligned}
\left|\left|\left|\widehat{T}_{S_1,S_2}(X)\right|\right|\right| &\le \left|\left|\left|\int_\Omega S_{1t} X S_{2t}\, d\mu(t)\right|\right|\right|_\Phi \\
&\le \left|\left|\left|\int_\Omega S_1 |X| S_1^*\, d\mu\right|\right|\right|_\Phi^{\frac{1}{2}} \left|\left|\left|\int_\Omega S_2^* |X| S_2\, d\mu\right|\right|\right|_\Phi^{\frac{1}{2}} .
\end{aligned}$$

Since $|||\cdot|||_{\Phi}$ is a dual norm related to a symmetric gauge function, the 2-convexoid norm $|||\cdot|||_{\Phi}^{(2)}$ is induced by

$$|||A|||_{\Phi}^{(2)} = |||A^*A|||_{\Phi}^{\frac{1}{2}},$$

for all $A^*A \in B(H_1, H_2)_{\Omega}$. Hence, (5) follows from (4), while (6) follows from (4) and (5), together with the duality argument applied to $\widehat{T}_{S_1,S_2}$. Here, $\widehat{T}_{S_2,S_1} : B(H_2) \to B(H_1)$ is the conjugate operator, and the corresponding transformers share a common norm. □

Theorem 2. *Let $\widehat{T}_{S_1,S_2} \in \mathfrak{B}_{\mathrm{IPTIT}}(B(H_1, H_2))$ and $S_1, S_2 \in B(H_1, H_2)$. If*

$$\int_{\Omega} S_1 S_1^* \, d\mu \quad \text{and} \quad \int_{\Omega} S_2^* S_2 \, d\mu$$

belong to $C_p(H)$ for some $1 \le q \le \infty$, then

$$\begin{aligned}
\left\| \widehat{T}_{S_1,S_2} \right\|_{B(H_1)\to B(H_2)} &\le \left\| k^{\frac{1}{2}} \left(\int_{\Omega} S_1 S_1^* \, d\mu \right) k^{\frac{1}{2}} \left(\int_{\Omega} S_2^* S_2 \, d\mu \right) \right\|_{l_q} \\
&\le \sqrt{\left\| \int_{\Omega} S_1 S_1^* \, d\mu \right\|_q \left\| \int_{\Omega} S_2^* S_2 \, d\mu \right\|} \\
&= \|S_1\|_{2q} \|S_2\|_{2q}.
\end{aligned}$$

Moreover,

$$\left\| \widehat{T}_{S_1,S_2} \right\|_{B(H_1)\to B(H_2)} \le \sum_{i=1}^{\infty} k^{\frac{1}{2}} \left(\int_{\Omega} S_1^* \left(\int_{\Omega} S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu \right) \left(\int_{\Omega} S_2 \left(\int_{\Omega} S_2^* S_2 \, d\mu \right)^{q-1} S_2^* \, d\mu \right).$$

In particular, if $S_2^* = S_1$, then

$$\left\| \widehat{T}_{S_1,S_1^*} \right\|_{B(H_1)\to B(H_2)} = \left\| \widehat{T}_{S_1,S_1^*}(I) \right\|_q = \left\| \int_{\Omega} S_1 S_1^* \, d\mu \right\|_q.$$

Proof. Consider the case $\alpha = q$. Then

$$\int_{\Omega^n} S_1 X S_2 \, d\mu \in C_p(H), \quad \forall X \in B(H).$$

Therefore,

$$\begin{aligned}
\|T_{S_1,S_2}(X)\| &\le \left\| \int_{\Omega} S_1 X S_2 \, d\mu \right\|_q \\
&\le \left\| \sqrt{\int_{\Omega} S_1 S_1^* \, d\mu} \right\|_q \left\| \sqrt{\int_{\Omega} S_2^* X^* X S_2 \, d\mu} \right\|_{2q} \\
&\le \left\| \sqrt{\int_{\Omega} S_1 S_1^* \, d\mu} \right\|_q \left\| \sqrt{\int_{\Omega} S_1 S_1^* \, d\mu} \right\|_q \|X\|.
\end{aligned}$$

To establish the last part of the theorem, observe that the range inclusion is similar to the trace of positive operators. In particular,

$$\int_{\Omega} S_1^* \left(\int_{\Omega} S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu$$

coincides with

$$\operatorname{tr} \left(\int_{\Omega} S_1 S_1^* \, d\mu \right)^q.$$

Thus, we have

$$\begin{aligned}
&\left\| \left(\int_\Omega S_1^* \left(\int_\Omega S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu \right)^{\frac{1}{2q}} X \left(\int_\Omega S_2 \left(\int_\Omega S_2^* S_2 \, d\mu \right)^{q-1} S_2^* \, d\mu \right)^{\frac{1}{2q}} \right\| \\
&\leq \left\| k_i^{\frac{1}{2q}} \left(\int_\Omega S_1^* \left(\int_\Omega S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu \right)^{\frac{1}{2q}} k_i^{\frac{1}{2q}} \left(\int_\Omega S_2 \left(\int_\Omega S_2^* S_2 \, d\mu \right)^{q-1} S_2^* \, d\mu \right)^{\frac{1}{2q}} \right\|_q \|X\| \\
&= \left\| \sqrt[q]{\sum_{i=1}^{\infty} k_i^{\frac{1}{2q}} \left(\int_\Omega S_1^* \left(\int_\Omega S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu \right) k_i^{\frac{1}{2q}} \left(\int_\Omega S_2 \left(\int_\Omega S_2^* S_2 \, d\mu \right)^{q-1} S_2^* \, d\mu \right)^{\frac{1}{2q}}} \right\|_q \|X\|,
\end{aligned}$$

which gives

$$\sum_{i=1}^{\infty} k^{\frac{1}{2}} \left(\int_\Omega S_1^* \left(\int_\Omega S_1 S_1^* \, d\mu \right)^{q-1} S_1 \, d\mu \right) \left(\int_\Omega S_2 \left(\int_\Omega S_2^* S_2 \, d\mu \right)^{q-1} S_2^* \, d\mu \right).$$

If $S_2 = S_1^*$, then

$$\begin{aligned}
\|\widehat{T}_{S_1,S_1^*}\|^q &\leq \sum_{i=1}^{\infty} K_i \left(\int_\Omega S_1^* (S_1 S_1^* \, d\mu)^{q-1} S_1 \, d\mu \right) \\
&= \operatorname{tr} \left(\int_\Omega S_1^* (S_1 S_1^* \, d\mu)^{q-1} S_1 \, d\mu \right) \\
&= \left\| \int_\Omega S_1 S_1^* \, d\mu \right\|_q^q \\
&= \|\widehat{T}_{S_1,S_1^*}(I)\|_q^q \\
&\leq \|\widehat{T}_{S_1,S_1^*}\|^q \\
&= \left\| \int_\Omega S_1 S_1^* \, d\mu \right\|_q .
\end{aligned}$$

□

Remark 1. The inequalities obtained here are established in a general Banach space setting. It is useful to consider special cases for emphasis and illustration.

4. Conclusion

Studies on inner product type integral transformers have been considered in many research works from various perspectives, including spectra, numerical ranges, and operator inequalities. An open problem remains concerning inequalities related to norm estimates for inner product type integral transformers whose spectra are contained in the unit disc. It has been observed that the norms of such transformers can be attained under the condition that one of the implementing operators is normal. In this note, we have addressed this problem by establishing norm inequalities for inner product type integral transformers in a general Banach space setting.

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