CHARACTERIZATION OF NORM ATTAINING OPERATORS IN C*-ALGEBRAS

SABASI OMAORO

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JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY

Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

SABASI OMAORO

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Signature .... Date ....

This thesis has been submitted for examination with our approval as the university supervisors.

1. Prof. Benard Okelo

Department of Pure and Applied Mathematics

Jaramogi Oginga Odinga University of Science and Technology, Kenya

Signature .... Date ....

2. Prof. Omolo Ongati

Department of Pure and Applied Mathematics

Jaramogi Oginga Odinga University of Science and Technology, Kenya

Signature .... Date ....

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Dedication

To my wife, Winfridah, and my children, Christine Kerubo, Beveline Moraa, Faith Nyasuguta, Boniface Ogero and Violet Nyaboke.

ABSTRACT

Various mathematicians have studied the subject of norm-attaining operators since its inception. Many forms of numerical ranges have been established which include the essential and the joint numerical ranges. However numerical ranges of norm-attaining operators in C*-algebras have not been fully investigated. Spectra of various operators have been studied and used to characterize other operators. However, spectra of norm attaining operators in C*-algebras are interesting and have not been fully investigated. Norms of various operators like elementary operators and others have also been studied over time by several authors and various results have been established. However in C*-algebras, norms of norm-attaining operators still remain interesting to mathematicians. This study focused on characterizing norm-attaining operators in C*-algebras. The specific objectives were to: characterize numerical ranges of norm attaining operators in C*-algebras, characterize spectra of norm-attaining operators in C*-algebras and establish norms of norm-attaining operators in C*-algebras. The methodology involved fundamental theorems like the Riesz Representation and Toeplitz-Hausdorff Theorems to characterize numerical range. In addition we employed inequalities such as Cauchy-Schwarz, triangle inequality and Polarization identity in establishing the norms. The results obtained show that; the numerical range of a norm-attaining operator S is non-empty and is equal to the convex hull of its point spectrum. In addition, the spectra is bounded and closed. Lastly, $\|S_o + S_1\| = \|S_o\| + \|S_1\|$ is equivalent to $\|S_o\|\|S_1\| \in \overline{W(S_o^*S_1)}$ where $S_o, S_1 \in NA(H)$. The results obtained are contributions of knowledge to C*-algebras and operator theory and a motivation to a further research. They may also be useful in mathematical formulation of quantum mechanics.

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INTRODUCTION

1.1 Mathematical background

Studying norm-attaining operators was influenced by Bishop-Phelps [30] Theorem where collections of norm-attaining functionals were shown to be dense in their duals. It is at this stage when Bishop and Phelps [30] raised the question on the possibility of extending the result to operators. Lindenstrauss [109] sought to answer the question by initiating a study where a first counter example was given and many other positive results were established. At the same time, Lindenstrauss introduced the two properties known as property A and property B where given two Banach spaces X_1 and Y_1 with $B(X_1, Y_1)$ denoting the algebra of bounded linear operators from X_1 into Y_1 and $NA(X_1, Y_1)$ denoting the set of norm-attaining operators, then X_1 satisfies property A in the case where $NA(X_1, Y_1)$ is found to be dense in $B(X_1, Y_1)$ in as much as all spaces Y_1 are concerned and Y_1 satisfies property B in the event that $NA(X_1, Y_1)$ is dense in $B(X_1, Y_1)$ in as much as each X_1 is concerned. For X_1 to have property B, Lindenstrauss [109] established that there must exist a set $\{x_\alpha, f_\alpha : \alpha \in A\}$ with $x_\alpha \in X_1, f_\alpha \in X_1^*$ and $\lambda < 1$ such that (i). $\|f_\alpha\| = 1$ and $\|x\| = \sup_\alpha |f_\alpha(x)|$ for each α and for all $x \in X_1$,

(ii). $\|x_\alpha\| = f_\alpha(x_\alpha) = 1$ for each α and $|f_\alpha(x_\beta)| < |\lambda|$ for all $\alpha \neq \beta$.

Examples of spaces which satisfy the above conditions include l_∞, C_0 and finite-dimensional spaces whose unit balls are polyhedral. The Bishop-Phelps [30] result was found not to be true for some normed spaces where on the interval $[0, 1]$, the space of real polynomials having supremum norm was cited as one of those which fail to satisfy the inference of the Bishop-Phelps Theorem [31]. Later Bishop and Phelps [29], gave a positive result for Banach spaces. Bollobás [33] obtained a quantitative version result by applying it to a problem of the numerical range of an operator and showed that for a Banach space X_1, with $x \in S_{X_1}$, $x^* \in S_{X^*}$ such that $|x^*(x) - 1| \leq \frac{\varepsilon^2}{2}$ for some $0 < \varepsilon < \frac{1}{2}$, then there are elements $y \in S_{X_1}$ and $y^* \in S_{Y^*}$ such that $y^*(y) = 1$, $\|y - x\| \leq \varepsilon$ and $\|y^* - x^*\| \leq \varepsilon + \varepsilon^2$. These results triggered an intensive interest on the subject with researchers such as Acosta [3] generalizing Lindenstrauss's [109] result for rotund spaces isomorphic to C_0 and Gowers [72] who, for l_p $(1 < p < \infty)$, showed that no infinite-dimensional Banach space with a strictly convex norm satisfies Lindenstrauss's [109] property B. But later Acosta et al [2] gave a new property to be satisfied by a Banach space if it has to have this characteristic where the collection of norm-attaining operators from any other Banach space X_1 into Y_1 is dense including for the finite-dimensional case.

Schachermayer [137] constructed an operator whose operator norm is unlikely to be approximated using norm-attaining operators. Such an operator is from $L^1[0, 1]$ to $C[0, 1]$ and is in the form of a pair of classical Banach spaces with the norm-attaining operators which are not dense. For spaces which satisfy property B, $C[0, 1]$ was cited as one which fails this property. Johnson and

Wolfe [87] had wondered on whether all Banach spaces could be made to have property B by renorming them. The question was answered by Partington [127] when he established that a norm $|||.|||_X$ with property B exists for a Banach space $(X, \|.\|)$ such that $\|\xi\| \leq |||\xi||| \leq c|\xi|$ for any $c > 3$ and $\xi \in X$.

Since then, many developments on the topic have taken place and many researchers have obtained various results and extensions to many other areas connected to the ideas of norm-attaining operators. These researchers include Bourgain [39] who established similarities between Radon-Nikodym and Bishop-Phelps properties for Banach spaces. Similar methods were employed to demonstrate that functionals of a subset C of a Banach space X exhibiting strongly exposing properties form the dual's dense G_δ subset on condition that C is bounded, closed and convex having all its non-empty subsets dentable. These notions of dentability of subsets for a Banach space and the Radon-Nikodym theorem for measures whose values are Banach spaces were introduced by Rieffel [134]. Following this, Maynard [113] and Davis and Phelps [54] showed that the Banach spaces for which the Radon-Nikodym property is true are the ones with dentable bounded and closed convex set. Later after observing that spaces with the Radon-Nikodym property such as reflexive spaces, separable spaces, among others, appear to be similar to those with the property that each bounded closed convex subset is equal to its extreme points' closed convex hull (Krein-Milman property), Diestel [56] sought to know whether the two properties were the same. Responding to this, Lindenstrauss [108] showed that indeed the property of Radon-Nikodym imply the property of Krein-Milman [24].

Schachermayer [137, 136] defined property α and β, two geometric concepts of Banach spaces, which generalize the geometric aspects of l_∞ and C_0. These are properties which had been used by Lindenstrauss [109] and Partington [127] while studying norm-attaining operators. Partington [127] showed that each Banach space may $(3+\epsilon)$-similarly be renormed to satisfy property β. Schachemayer [136] followed up on this to show that many Banach spaces may $(3+\epsilon)$-equivalently be renormed to have property α which is like a predual version of Partington's [127] version applying to an enormous class of Banach spaces. While Finet and Paya [66] established the density of $NA(L_1(\beta), L_\infty[0,1])$ in $B(L_1(\beta), L_\infty[0,1])$ for each σ-finite measure β which gave a new example of Hausdorff space K which is compact with $NA(L_1[0,1], C(K))$ being approximated by $B(L_1[0,1], C(K))$. Paya and Saleh [129] later extended this result to show that given any arbitrary measure β and any measure M which is localizable, an assurance is given for the set of norm-attaining operators to be dense in $B(L_1(\beta), L_\infty(M))$.

Alaminos et al [9] gave a condition which is sufficient for a C^*-algebra to guarantee that a weakly compact operator into a Banach space is dense in the set of norm attaining operators and that each continuous bilinear form is also dense in the set of norm-attaining bilinear forms. Uhl [158] proved that a norm-attaining operator is norm dense if any convex Banach space Y satisfies Radon-Nikodym condition. Lee [104] looked at paranormal operators which attain their norms and established that they contain nontrivial invariant subspaces. Miguel [116] found out that between Banach spaces there are linear operators which, though compact, cannot be approximated by norm-attaining operators. Many researchers studying norm-attaining operators between Banach spaces have emphasized on the more general question about the denseness property being satisfied by any collection of these operators.

Quadratic forms and their related properties motivated the study of numerical ranges of operators which has elicited much interest from various researchers. The concept is traced to Toeplitz [155] who is credited to have defined what was called by then as "the field of values for a matrix". It was soon found that this concept could be extended to the study of bounded linear operators acting on a Hilbert space. Miguel [115], using quadratic forms, interpreted the numerical range to be the range of $\widehat{\phi(S)}$ restricted to the unit sphere of H where $\phi(S)\langle x, y\rangle = \{\langle Sx, y\rangle : x, y \in H\}$ and $H\widehat{\phi S}(x) = \phi S\langle x, x\rangle = \{\langle Sx, x\rangle : x \in H\}$. It is indicated that emphasis is put on the image of the unit sphere because it can easily be used to describe the unit ball's image and that of the entire range but not the other way round. Algebraic and geometrical characteristics of the unit were established by Bohnenblust [32], specifically that the unit ball's algebras has the unit as its vertex. This finding became useful in constructing characterizations for C*-algebras. Lumer [110] and Bauer [20] followed this trend closely by extending the idea of an operator on a Banach space's numerical range to spaces not using their algebraic structures. More so the result by Lumer [110] which establishes that the numerical range of an operator is always bounded by the norm of the operator has greatly influenced studies on the theory.

Marcus and Pesc [111] investigated numerical ranges of square complex matrices and established that those which are 3×3 or 4×4 strictly upper triangular with real entries have their numerical ranges being circular discs with the origin as center. Chien and Tam [44] extended this result to an arbitrary 3×3 complex or 4×4 real upper triangular matrix. Later on Keeler et al [90] characterized a 3×3 complex matrix that has an elliptical disk and also one that has a circular disk as its numerical range. Similarly, Li and Tsing [106] also established equivalent conditions on a complex matrix all of whose C-numerical ranges are invariant under all rotations about the origin of the plane. Characterizations for a matrix whose C-numerical ranges have weak circular symmetry were obtained and it was found that under rotation through the angle $\frac{2\pi}{g}$ (g an integer greater than one) and center the origin, the numerical ranges are invariant.

Theoretical study and applications have motivated researchers to consider different generalizations of the numerical range. These include Halmos [76], Horn and Johnson [81] who have dedicated whole chapters to the subject in their works and Gustafson and Rao [73] with a whole book on the same. As stated by Chi-kwong and Yiu-Tung [43], the *Davis Wielandt* shell for $S \in B(H)$ given as $DW(S) = \{(\langle Sy, y\rangle, \langle Sy, y\rangle) : y \in H, \langle y, y\rangle = 1\}$ is among the most influential generalizations in analysis. It is noted that the set $DW(S)$ projected on the first coordinate is equivalent to the classical numerical range of the operator S and determination of its interior can easily be done despite the fact that its boundary points might not be understood so well.

The numerical range, like the spectrum, being a subset of the complex plane has geometrical properties which can give certain insights into the operator involved. For instance it is a well known fact that if the numerical range of an operator is a subset of the set of real numbers then it implies that the operator must be self-adjoint. But having similar knowledge about the spectrum of an operator tells nothing meaningful about that operator. For the numerical range, both its norm and algebraic properties can be drawn out from its definition especially about its closure containing the operator's spectrum and its norm being equal to at most twice its numerical radius gives algebraic properties of the operator in addition to properties about its norm. It is almost one hundred years since the study on numerical ranges of linear operators which are bounded was started as stated by Hwa-Long [82], starting with the amazing result of Hausdorff-Toeplitz [78, 155] which established the numerical range's convexity in the complex plane implying that H's unit $\xi = 1$ is mapped to the complex plane's subset by the quadratic form g, with a full interior.

Over time, many generalizations about the numerical range have been realized depending on various contexts. For example the two works by Bonsall and Duncan [34, 35] are results of some analysts in Britain after conducting studies based on normed algebra elements. The subject has been intertwined to many areas including in iteration processes, matrix polynomial factorizations, unitary similarities among others as found in Mecheri [114], Axelson et al [19], Horn and Johnson [81] and Istratescu [83].

Toeplitz [155] defined the numerical range on spaces which are finite-dimensional after being influenced by earlier studies on the quadratic form's classical theory. The containment of all the eigenvalues of an operator on finite-dimensional Hilbert space in its numerical range is a well established result which has influenced many other results in the Banach space setting. Toeplitz [155] demonstrated the convexity of the boundary of the numerical range's complement and Haursdorf [78] established the convexity of $W(S)$ in Hilbert spaces which are finite-dimensional. Similarly, Stone [149] found out that for pre-Hilbert spaces picked arbitrarily, convexity of any operator's numerical range in these spaces is guaranteed.

Much has been done about the numerical range together with an indication on the many areas where it is applied. Authors who have worked on this include Berger and Stampfli [26] who dealt with the inequality $w(S^n) \leq (w(S))^n$ which marked a significant achievement since it had been established how untrue

$w(ST) \leq w(S)w(T)$ was. What it means to have the origin in the numerical range or on its boundary has been examined by some researchers. This is what Embry [63] did when considering the relation $KA = AH$ and established that $K = H$ implies that $0 \notin W(A)$ whenever K and H are normal commuting operators. Khalagai [92] while studying conditions of invertibility in partial isometries, showed that S is unitary whenever it is a partial isometry and $0 \notin W(S)$ or $0 \notin W(S^2)$. Paul and Shapiro [141] considered Hardy spaces, H^2, and established that composition operators defined on these spaces, not the identities, have numerical ranges which contain the origin in their closures and contain the origin if and only if they are closed.

It is indicated in Enders [60] that it is Sir Isaac Newton who introduced the term "spectrum" which meant image in Latin language. von Neumann and more other mathematicians picked up the subject and it has been studied over time and used to characterize many operators.

Letting J represent a self-adjoint involution on H, Albeveiro et al [10] constructed an operator on H and let it to be $L = B + V$ having B, a self-adjoint operator, being unbounded and commuting with J while V being bounded and self-adjoint but non-commuting with J. Optimal estimations on the the spectrum of L were established in relation to the spectrum of B obtaining bounds for the norm of the operator.

Damak and Jeribi [51] investigated the essential spectrum of matrix operator on Banach spaces, particularly that of order 2×2 taking the form $L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Results were obtained which were then used to describe the essential spectra of differential operators.

Norms of various operators have been studied over time by many mathematicians and various results have been established. Khalid and Albadawi [93] used known inequalities to obtain sharp absolute value inequalities for norms of operators. Such inequalities include Cauchy-schwarz and Minkowski type. Okelo et al [124] studied the operator $U_{A,B(X)} = AXB + BXA$, which is the Jordan Elementary operator and determined, for each $X_1 \in B(H)$, its norm for B and A fixed in $B(H)$ for each $X \in B(H)$. Specifically, the inequality $\|U_{A,B}\| \geq \|A\|\|B\|$ was established and proved. Calculating the exact value of $\|T\|_{p,q}$ for positive linear operators between L^p spaces was shown, by Ralph and Anton [133], to be equivalent to determining the existence of a non-negative solution for some nonlinear functional equation through computation of the exact value for $\|W\|_{p,q}$ of the operator $Wg(t) = \int_0^t g(x)dx$.

Bauer et al [22] observed that the diagonal elements' maximum moduli is equal to the least upper bound of a diagonal matrix . They characterized these sets of supremum norms and norms of the vectors in which they belong. Properties satisfied by these norms were also given which led to the introduction of what they called absolute norms and monotonic norms for vectors. Earlier, Bauer and Fike [21] had stated and proved some theorems relating to the characteristic roots of two matrices where for x and λ being a characteristic vector and corresponding characteristic root respectively and A being an arbitrary matrix, then either $x(\lambda I - A)^{-1}(B - A)x$ or λ is A's characteristic root giving $\|x\| \leq lub[(\lambda I - A)^{-1}(B - A)]\|x\|$.

Kumar and Sinclair [98] considered projective norms for operator and Banach spaces and demonstrated that they are equivalent for $B \otimes C$ on condition that B and C are C^*-algebras which are subhomogeneous.

Choi and Li [45] obtained a result which showed that the triangle inequality gives the upper bound norm estimate for two operators' sum given $\sup\{\|C^*AC + U^*BU\| : C, U\ unitary\}$ is equal to $\min\{\|A+\beta I\|+\|B-\beta I\| : \beta \in \mathbb{C}\}$. It was established that this relation is useful in characterizing norms which are unitarily invariant and can be expressed as finite-dimensional. Stampfli [148] considered the inner derivation $\mathfrak{D}_S : A \to SA - AS$ on $B(H)$ and determined its norm. It is in this work where it was established that $\|\mathfrak{D}_S\| = \inf\{2\|S - \lambda I\| : \lambda \in \mathbb{C}\}$ if S is normal and that $\|\mathfrak{D}_S\|$ can be specified in $\sigma(T)$ geometric terms.

Estimates of Hardy-Little wood for operator norms on sequence spaces were considered by Osikiewicz and Tonge [125] where interpolation theory approaches were used in proving results in such areas. Also Timoney [153] proved directly that in $B(H)$, whenever an elementary operator's norm is to be estimated, it is best done by using a generalization of Stampfli's Theorem. It was shown that given an elementary operator S whose length is L whenever $k = L$, then the k-norm's value and the completely bounded norm are equal.

Crouzeix [50] considered numerical ranges of square matrices involving polynomial functions of these matrices and proved an inequality related to the polynomial functions. Valid extensions for operators, both bounded and unbounded, were also shown which resulted in the establishment of a functional calculus.

1.2 Basic concepts

Definitions of concepts relevant to this study are reviewed in this part and results on numerical range, numerical radius, norms, spectrum, operator, C*-

algebras and others which are relevant to this study are also given. We also give some examples and remarks on certain concepts.

Definition 1.1 ([12], Definition 0.4.1)**.** Let V be a vector space. Then the map $\langle ., . \rangle : V \times V \to \mathbb{K}$ is called an inner product on V if $\forall\ \xi,\ \zeta,\ \eta \in V$ and $\lambda \in \mathbb{K}$ the properties below are satisfied:

(i). $\langle \xi, \xi \rangle \geq 0$ with $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

(ii). $\langle \xi + \zeta, \eta \rangle = \langle \xi, \eta \rangle + \langle \zeta, \eta \rangle$.

(iii). $\langle \lambda\xi, \zeta \rangle = \lambda \langle \xi, \zeta \rangle$.

(iv). $\langle \xi, \zeta \rangle = \overline{\langle \zeta, \xi \rangle}$.

V together with $\langle ., . \rangle$ is called an inner product space.

Definition 1.2 ([166], Section 4.4)**.** Let V be a vector space. A nonnegative function $\|.\| : V \to \mathbb{R}$ with real values is a norm if the following conditions are satisfied:

(i). $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0\ \ \forall x \in V$.

(ii). $\|x + y\| \leq \|x\| + \|y\|\ \ \forall x, y \in V$.

(iii). $\|\lambda x\| = |\lambda| \|x\|\ \ \forall \lambda \in \mathbb{C}$ and $x \in V$.

$(V, \|.\|)$ is then known as a normed space.

Definition 1.3 ([12], Definition 0.7.1)**.** If every Cauchy sequence in a normed space V is convergent (i.e, it has a limit in V) then V is known as a *complete* space .

Definition 1.4 ([12], Definition 0.7.2). A normed space which is complete is called a Banach space.

Definition 1.5 ([12], Definition 0.7.3). A normed inner product space which is complete is called a Hilbert space.

Definition 1.6 ([12], Definition 0.6.2). A separable space is a normed linear space which contains a countable dense subset.

Definition 1.7 ([53], Definition 3.2). Let U be an operator in H such that $U^*U = UU^* = I$, with U^* being the adjoint of U and $I : H \to H$ being an identity operator. Then U is called a unitary operator.

Definition 1.8 ([114], Definition 2.4). Let S be a linear operator. Then the set $W(S) = \{\langle S\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}$ in the complex plane is called the numerical range of S.

Definition 1.9 ([52], Definition 3.6). An operator $S \in B(H)$ is said to be compact if for each bounded sequence of vectors $(x_n) \in H$, the sequence $\{Sx_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Definition 1.10 ([16], Definition 4). Let $\mathfrak{B}$ be a Banach algebra. Then $\mathfrak{B}$ is called a C^*-Algebra if it has an involution $*$ such that for all $b \in \mathfrak{B}$, $\|b^*b\| = \|b\|^2$ is true.

Definition 1.11 ([165], Definition 2.2). Let S be an operator. The spectrum of S is denoted and given by $\sigma(S) = \{\lambda \in \mathbb{C} : (S - \lambda I) \text{ does not have an inverse}\}$.

Definition 1.12 ([165], Definition 2.1). The resolvent set of an operator $S \in B(H)$ is the set $\rho(S) = \{\lambda \in \mathbb{C} : (S - \lambda I) \text{ is invertible}\}$ i.e $\rho(S) = \mathbb{C} \backslash \sigma(S)$.

Definition 1.13 ([38], Definition 4.4). The point spectrum of S consists of all the eigenvalues $\lambda \in \sigma(S)$ such that $(S - \lambda I)$ is not one-to-one. It is also called the discrete spectrum.

Definition 1.14 ([144], Definition 2.3)**.** The continuous spectrum of S consists of all $\lambda \in \sigma(S)$ such that $(T - \lambda I)$ is one-to one but not onto and $\operatorname{ran}(S - \lambda I)$ is dense in H.

Definition 1.15 ([144], Definition 2.5)**.** The residual spectrum of T consists of all $\lambda \in \sigma(T)$ such that $(T - \lambda I)$ is one-to-one but not onto with $\operatorname{ran}(T - \lambda I)$ not being dense in H.

Definition 1.16 ([165], Definition 3.1)**.** The Spectral radius of an operator S is given by: $r(T) = \sup\{|\lambda| : \lambda \in \sigma(S)\}$.

Definition 1.17 ([136], Section 1)**.** $S \in B(H)$ is said to be norm-attaining if there is $\xi \in H$ with $\|\xi\| = 1$ such that $\|S\xi\| = \|S\|$. S here is said to attain its norm at this ξ.

Remark 1.18. In a Hilbert space, the set of all norm-attaining operators is denoted by $NA(H)$.

Norm attaining operators on a Hilbert space H form a special class of C^*-algebras as shown below:

Proposition 1.19. *Let H be a Hilbert space, then $NA(H)$, the algebra of norm-attaining operators on H, is a $*$-algebra.*

Proof. Let $S_1, S_2 \in NA(H)$ and $\lambda \in \mathbb{C}$. Then for $\forall \xi, \eta \in H$ we have

(i) $\langle S_1^{**}\xi, \eta\rangle = \langle \xi, S_1^*\eta\rangle = \langle S_1\xi, \eta\rangle. \Rightarrow S_1^* = S_1.$

(ii) $\langle (S_1S_2)^*\xi, \eta\rangle = \langle \xi, S_1S_2\eta\rangle = \langle S_1^*\xi, S_2\eta\rangle = \langle S_2^*S_1^*\xi, \eta\rangle$
$\Rightarrow (S_1S_2)^* = S_2^*S_1^*.$

(iii) $\langle(\lambda S_1 + S_2)^*\xi, \eta\rangle = \langle\xi, (\lambda S_1 + S_2)\eta\rangle$

$= \langle\xi, \lambda S_1\eta\rangle + \langle\xi, S_2\eta\rangle$

$= \overline{\lambda}\langle\xi, S_1\eta\rangle + \langle\xi, S_2\eta\rangle$

$= \overline{\lambda}\langle S_1^*\xi, \eta\rangle + \langle S_2^*\xi, \eta\rangle$

$= \langle(\overline{\lambda}S_1^* + S_2^*)\xi, \eta\rangle \Rightarrow (\lambda S_1 + S_2)^* = \overline{\lambda}S_1^* + S_2^*$

This implies that $NA(H)$ is a $*$-algebra. □

Proposition 1.20. *Let H be a Hilbert space, then $NA(H)$ is a Banach algebra.*

Proof. Let $S_1, S_2 \in NA(H)$ and ξ. Then we have

$\|S_1S_2\| = \sup\{\|(S_1S_2)\xi\| : \xi \in H, \|\xi\| = 1\} \leq \sup\{\|S_1\|\|S_2\xi\| : \|\xi\| \leq 1\} = \|S_1\|\|S_2\|$

$\Rightarrow NA(H)$ is a Banach algebra. □

Proposition 1.21. *Let H be a Hilbert space, then $NA(H)$ is a C^*-algebra, that is $\|S^*S\| = \|S\|^2$.*

Proof. Let $S \in NA(H)$ and $\xi \in H$. Then we have

$\|S\|^2 = \sup\{\|S\xi\|^2 : \xi \in H, \|\xi\| \leq 1\}$

$= \sup\{\langle S\xi, S\xi\rangle : \xi \in H, \|\xi\| \leq 1\}$

$= \sup\{\langle S^*S\xi, \xi\rangle : \xi \in H, \|\xi\| \leq 1\}$

$\leq \sup\{\|S^*S\xi\|\|\xi\| : \xi \in H, \|\xi\| \leq 1\}$

$= \sup\{\|S^*S\xi\| : \xi \in H, \|\xi\| \leq 1\} = \|S^*S\|$

$$\Rightarrow \|S\|^2 \leq \|S^*S\| \tag{1.2.1}$$

Also, $\|S^*S\|^2 = \sup\{\|S^*S\xi\|^2 : \xi \in H, \|\xi\| \leq 1\}$

$= \sup\{\langle S^*S\xi, S^*S\xi\rangle : \xi \in H, \|\xi\| \leq 1\}$

$= \sup\{\langle S\xi, S\xi\rangle : \xi \in H, \|\xi\| \leq 1\}$
$= \sup\{\|S\xi\|^2 : \xi \in H, \|\xi\| \leq 1\}$
$\leq \sup\{\|S\|^2\|\xi\|^2 : \xi \in H, \|\xi\| \leq 1\} = \|S\|^2\}$

$$\Rightarrow \|S^*S\| \leq \|S\|^2 \tag{1.2.2}$$

From inequality 1.2.1 and 1.2.2 it follows that $\|S^*S\| = \|S\|^2$. □

Remark 1.22. In this study $NA(H)$ is taken to be a C^*-algebra unless otherwise stated.

1.3 Statement of the problem

Properties of norm-attaining operators have been studied by several mathematicians in various algebras for instance Calkin algebra, standard operator algebra, Banach algebras among others. However, these properties have not been discussed exhaustively. Kingangi [94] studied numerical ranges of operators in C*-algebras and showed that zero is in an operator's algebraic numerical range if and only if the operator is orthogonal to the identity operator. Additionally, it has been shown that the spatial numerical range's closure is equal to the operator's algebraic numerical. However in this work, it has been recommended that a similar study be done on the spectra of operators in C*-algebras. Hence in our work, we have studied norm-attaining operators in C*-algebras and because numerical ranges and spectra for operators are closely related, we have therefore characterized their numerical ranges and spectra accordingly. At the same time, Okelo [122] worked on norms and norm-attainability for normal operators and obtained norm attainability conditions for Hilbert space operators. However in this work, a question was posed as follows: What are the norms of these operators if the norm attainability condition suffices?. Similarly in Okelo [121], characterization of norm-attainability and orthogonality of elementary operators in H was done where necessary and sufficient conditions were given for norm-attainability of Hilbert space operators and range kernel orthogonality of elementary operators. It was, however, recommended that further extension be done on generalized finite operators in C*-algebras. Therefore, in this study we have established the norms of norm-attaining operators in C*-algebras via the lower bound estimates and upper bound estimates mirroring Martin Mathieu's open problem [112] which requires one to obtain norms of elementary operators in a general Banach-algebra setting.

1.4 Objectives of the study

1.4.1 Main objective

The main objective of this study is to characterize norm attaining operators in C^*-algebras.

1.4.2 Specific objectives

The specific objectives of the study include to:

(i). Characterize numerical ranges of norm-attaining operators in C*-algebras.

(ii). Characterize spectra of norm-attaining operators in C*-algebras.

(iii). Establish norms of norm-attaining operators in C*-algebras.

1.5 Significance of the study

The findings of the study are contributions to knowledge on C^*-algebras and operator theory. Moreover, they may be useful in applications to other areas for example dilation theory, Krein space operators, factorizations of matrix polynomials and unitary similarity. In mathematical formulation of quantum mechanics and quantum mechanical systems, operators which attain their norms are very crucial for use by engineers in such fields as energy and architecture while designing structures like buildings, roads and bridges which can withstand pressure from various natural forces. In principal component analysis, eigenvalues are useful in applications to do with image processing. Several methods exist for processing of images for instance in measurement of sharpness, eigenvalues play a big role. In segmentation of human faces while creating their images, the smallest and largest covariance matrix eigenvalues are used. This can be used in security sector in face recognition to identify criminals.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

C*-algebras is a subject of study in functional analysis which has captured the interest of many mathematicians over several years. The study of various operators in these algebras have been done and still the subject keeps on attracting attention. In this chapter, we give a detailed review of related literature. We consider norm-attaining operators and their numerical ranges, spectra and norms.

2.2 Norm-attaining operators

Norm-attaining operators is a topic which has received significant attention by a number of mathematicians. Lindenstrauss [109] pioneered the study into these operators by demonstrating that there are Banach spaces which could be matched such that between them, bounded linear operators could be found having the property that they are not dense in the set of norm attaining operators. Domain and range conditions to be satisfied for norm attaining operators to be dense were outlined in this work. To confirm how dense norm attaining operators are, a result providing a constructive proof is given by Enflo [61] who established that for any linear and bounded operator B in H, there must exist some sequence K_m of operators whose rank is 1 and operator norm converges to zero so that for each m, $B + K_m$ attains its norm. Particularly, this construction confirmed the norm-attaining operators' norm denseness in $B(H)$. Similarly this construction was used for operator valued functions which attain their norm to show their denseness. A question was asked by Miguel [116] on whether $L(X_1, Y_1)$ could be used to approximate $NA(X_1, Y_1)$ for Banach spaces X_1 and Y_1. It was observed that this was an open problem including for real two dimensional Hilbert space Y_1. Later, Schachermayer [137] constructed an operator where norm-attaining operators may not approximate its operator norm. This was a response to that question asked by Johnson and Wolfe [88] and it became the first illustration where marching Banach spaces given fail to have norm-attaining operators being dense.

Alaminos et al [8, Theorem 2] shows spaces of continuous functions with bilinear forms which can be approximated by bilinear forms which attain their norm. It is established that every operator which is weakly compact from $C_0(L)$ into a real or complex Banach space can be approximated by weakly compact operators which attain their norm. The result depends on Schachermayer [136] who considered compact operators from $C(K)$ into other Banach spaces and established that they can be approximated by norm-attaining operators. The result was applied on spaces of continuous functions to get denseness of bilinear forms which are norm-attaining. This was confirmed by the following result:

Theorem 2.1. *([8], Theorem 3) All bilinear forms which are continuous on $C_0(L)$ can be approximated by norm-attaining bilinear forms.*

From Theorem 2.1 , it can generally be observed that for $N > 2$, denseness of norm-attaining N-linear forms on a Banach space is never implied by denseness of bilinear forms that attain their norms as proved by Sevilla and Paya [140, Proposition 2.1] who also established that norm-attaining quadratically hyponormal weighted shifts are subnormal.

Later Alaminos et al [9] obtained a result which gave sufficient C*-algebra conditions under which operators which are weakly compact are dense in the space of continuous linear operators which attain their norms. It is here where it is also clearly established that for each continuous linear operator from a Banach space into its dual there is an associated bilinear form on the same Banach space. In this result, [9, Theorem 2], the C*-algebra has been restricted to a condition that for each positive linear functional on the algebra there is an associated positive linear functional on the same algebra such that if there is a projection on the bi-dual of the algebra then there must be a corresponding central projection on the bi-dual of the algebra for the result to hold.

The preceding result indicates that for the bilinear form to be norm-attaining, the corresponding operator must also be norm-attaining, but the converse is not true as shown by Paya and Finet [130].

Having used a very lengthy and intricate example to show an operator $S_0 : L^1[0,1] \to C[0,1]$ which is such that if $\|S_0 - S\| \leq \frac{1}{2}$ with $1 \geq \|S\|$ then S is not norm-attaining and asking for a characterization of compact spaces K from where norm-attaining operators are dense in $B(L^1[0,1], C(K))$, Schachermayer [136] inspired Johnson and Wolfe [88] who identified a pair of a compact metric space A and an operator $S_0 : L^1[0,1] \to C(A)$ such that if $\|S_0 - S\| \leq \frac{1}{2}$ with $1 \geq \|S\|$ then S is not norm-attaining. The result was achieved through the following corollary:

Corollary 2.2. *([88], Corollary 2) Define* $S_0 : L^1[0,1] \to C(A)$ *by* $S_0 f(A) = \int_0^1 f(\mu)A(\mu)d\mu$ *for* $f \in L^1$ *and* $a \in A$*. If* $S : L^1[0,1] \to C(A)$ *is a linear operator with* $\|S_0 - S\| \leq \frac{1}{2}$ *and* $1 \geq \|S\|$ *then* S *is not norm-attaining.*

Compared to Schachemayer's example, Corollary 2.2 is simple and short in proof.

Basing his work on Partington's paper [127] where it is shown that renorming a Banach space can equivalently make it satisfy "property B", Schachemayer [137] defined, for a Banach space, two geometric concepts, property β and property α. Partington had shown that the renormings confirm a criterion called "property β" in which Lindenstrauss had shown that it implies property B. Here, Schachemayer gave a predual version of Partington's construction which applies to large classes of Banach spaces having an equivalent renorming confirming property A. He established the difference between the Bishop-Phelps property and property A.

Pandey and Vern [128] studied operators which attain their norm on all closed subspaces of arbitrary-dimensional complex Hilbert spaces and established a spectral characterization result for the operators in [128, Theorem 6.3] which depends on the polar decomposition theorem and the following Lemma for its proof:

Lemma 2.3. *([128], Lemma 6.2) Let* H_2 *and* K_2 *be complex Hilbert spaces and let* $S \in B(H_2, K_2)$*. Then* S *is* $\mathcal{AN}$ *if and only if* $|S|$ *is* $\mathcal{AN}$*.*

In investigating some properties of these operators the authors constructed the following example.

Example 2.4. ([128], Example 3.3) Let K_1, K_2 be positive compact operators on the complex Hilbert space l^2, and $0 \leq a < b$. Consider the operator

$$T = \begin{pmatrix} aI + K_1 & 0 \\ 0 & bI + K_2 \end{pmatrix} \in B(l^2 \oplus l^2)$$

It is claimed that this operator does not satisfy the property $\mathcal{AN}$.

Example 2.4 shows that, under addition, these operators are not closed. It was proved that a proper cone in the real Banach space of hermitian operators is formed by the intersection of the operators with positive operators.

In [87] the existence of norming functionals interested Johnson and Wolfe especially when considering $C(K)$ where in [87, Theorem 1] it was shown that $B(C(S), C(K))$ (with S and K being compact Hausdorff spaces) can be approximated by $NA(C(S), C(K))$. Using the property that the measure $U(\Sigma)$ is a $C(S)$ space isometrically and that, by Hahn Banach Theorem, the adjoint of a norm-attaining operator is also norm-attaining it was established that there exists a number of elements in $ca(\Sigma, C(K))$ which possess norming functionals but do not have relatively compact ranges.

While addressing the issue of a countably additive measure with Banach space values attaining its range's diameter, Russel and Lewis [135] gave an example of a vector measure which is countably additive with values in $C(K)$ for which the diameter of the range is not attained. Here, for infinite-dimensional measures into a space having smooth dual, a property stronger than the attainment of the diameter is shown to fail. A characterization of the countably additive measures into the space having smooth dual was given. In this paper, techniques from [87] were used to construct a measure μ which is countably additive whose values are in the space C_0 so that μ fails to attain its diameter.

Pellegrino and Teixeira [131] obtained results showing that for $1 < p < \infty$ and $1 \leq q < \infty$, a linear operator $T : \ell_p \to \ell_q$ will attain its norm if and only if a not weakly null maximizing sequence for T exists. Consequently for $1 < p \neq q < \infty$, they showed that any weakly null maximizing sequence for a norm-attaining operator $T : \ell_p \to \ell_q$ has a norm-convergent subsequence. They also investigated lineability of the sets of norm-attaining and non-norm-attaining operators.

Studying paranormal and hyponormal operators, Jun Ik Lee [104] established that norm-attaining paranormal operators always have nontrivial invariant subspaces and as a result norm-attaining hyponormal operators also have non-trivial subspaces.. The result is as follows:

Theorem 2.5. *([104], Theorem 3) If $T \in B(H)$ is a norm-attaining paranormal operator then T has a non-trivial invariant subspace, and hence norm-attaining hyponormal operators have non-trivial invariant subspaces.*

The property that H has closed subspaces which are nonempty is used to prove Theorem 2.5 besides the following basic property for norm-attaining operators.

Lemma 2.6. *([104], Lemma 1) $S \in B(H)$ is a norm-attaining operator if and only if $\|S\|^2 \in \sigma_p(S^*S)$, where $\sigma_p(S)$ is the point spectrum of $S \in B(H)$.*

2.3 Numerical ranges of Operators

Toeplitz [155] first defined the numerical range on a finite dimensional space after having been influenced by the study on quadratic forms, a theory which had become very influential at the time. Studies have established that in a finite-dimensional space, the numerical range contains the operator's eigenvalues, a result which influenced Toeplitz [155] to study and establish that for the complement of the operator's numerical range, its boundary is a convex curve. Similarly, Haursdorf [78] has proved the convexity of the numerical range of an operator in finite dimensional Hilbert spaces. It was also proved by Stone [149] that in any pre-Hilbert space, the numerical range of an operator is convex.

The Toeplitz-Hausdorf theorem in two dimensional case has been proved by Shapiro [141] and then extended to general spaces through projections. The same theorem confirms the convexity of the numerical range of an operator and the containment of the spectrum in the closure of the numerical range of the operator as shown by Gustafson and Rao [73]. It is here where it has been shown that T's convex hull is the same as the intersection of the numerical ranges' closures for such operators. Jahedi and Yousefi [86] studied reflexive Banach spaces and defined the numerical range in the spaces which possesses canonical numerical range's basic properties. Similarly in this work, composition operators on Hardy spaces were considered where necessary and sufficient conditions were determined for their numerical ranges' closures to be closed in addition to giving conditions under which zero is contained in the closures of those numerical ranges. The result is as follows:

Theorem 2.7. *([86], Theorem 3) Let* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\sum\limits_{m\geq 0} m^{qi}/\alpha(m)^p$ *is undefined for some* i*. Then* $0 \in \overline{W(C_\vartheta)}$ *if on* $H^q(\alpha)$*,* C_ϑ *is bounded.*

In Theorem 2.7 a consideration of composition operators on weighted Hardy spaces and general hardy spaces has been made.

An operator's numerical range and its norm were studied by Kittaneh [95] who established and proved the relationship $w(T) \leq \frac{1}{2}(\|T\| + \|T^2\|^{\frac{1}{2}})$ which connects the two. Miguel et al [117] while considering a bounded function g from a closed subspace V of a Banach space W's unit sphere sought to find out when the intrinsic numerical range of the function is equal to the spatial numerical range's closed convex hull. The findings were that there is a subspace V for every finite dimensional Banach space W together with a bounded linear operator $S : V \to W$ giving $\overline{co}W(T) \neq V(T)$. An infinite dimensional Banach space was considered since it is known that when X is finite-dimensional, for every superspace Y and every continuous function $f : S_X \to Y$, the equality $\overline{co}W(f) = V(f)$ holds [79]. It was also shown that up to renorming, for every non-reflexive Banach space Y, one can find a closed subspace X and a bounded linear operator $T \in L(X, Y)$ such that $\overline{co}W(T) \neq V(T)$. Additionally, a sufficient condition was introduced for the closed convex hull of the spatial numerical range to be equal to the intrinsic numerical range, which they called the Bishop-Phelps-Bollobas property and which is weaker than the uniform smoothness and the finite dimensionality. Here, strong subdifferentiability and uniform smoothness were characterized in terms of this property.

Donoghue [57] studied the numerical range of a bounded operator in which he gave a precise description of the numerical range for the special case where the Hilbert space H is two-dimensional. He used methods which provided an easy and natural proof that $W(T)$ is convex. He also gave alternative methods of proving that if $W(T)$ reduces to the single point λ, then $T = \lambda I$, where I is the identity and that if $W(T)$ is a subset of the real axis, then T is self-adjoint.

For some bounded operator, Radjabalipour and Radjavi [132] established that its numerical range can be any subset E of $\mathbb{C}$ where $E \setminus IntE = A_0 \cup A_1$ with A_0 countable and A_1 being a union of a conic section's smooth subarcs. Agler

[5] studied the numerical range's geometry together with that of its boundary, particularly $W(E) \setminus Int\ W(E)$. Lancaster [100] investigated the numerical range's boundary and brought about the essential numerical range proving results relating set properties of the numerical range's boundary and that of the essential numerical range, proofs which were later simplified by Williams [160].

H-Wa Long and Pei [82] surveyed three topics which were current at the time; namely a condition for a finite matrix's numerical range to be a circular disk by Anderson, the conjecture by Holbrook on the inequality of the numerical radius of two commuting operator's product on the structure Theorem by Williams and Crimmins [162] concerning when half of the norm of an operator is equal to its numerical range. It was shown that interchanging places for S_1 and S_2 where S_2 is of class S_n we have $w(S_1S_2) \leq w(S_1)\|S_2\|$ holding as long as S_1 and S_2 commute. The same inequality is also true when S_1 is a quadratic operator. However under the same conditions it is not clear whether $w(S_1S_2) \leq \|S_1\|w(S_2)$ is true or not. In this case, the authors established that the relationship is true whenever S_1 is idempotent or is square-zero.

Also Wu [164, Theorem 1] established that for a finite matrix represented as $\begin{pmatrix} aI & B \\ 0 & C \end{pmatrix}$, its numerical range is a disc which is circular with center at a if a is C's eigenvalue. As a consequence of this result and for any finite matrix S, it was found that $\partial W(S)$ does not have a circular arc if S is similar to a normal matrix but if the center is an eigenvalue of S then $\partial W(S)$ has a circular arc.

Gau and Wu [69, Theorem 1] established that $2n + 1$ is the least number of common supporting lines needed in order for an elliptic disc A to be contained in $W(S)$ where A is the intersection points of the supporting lines and the numerical range $W(S)$ of S. This is a generalization of results obtained earlier by Anderson and Thomson [132], applied to prove the conjecture of the poncelet property as a special case. At the same time in [70, Theorem 2.3] the interest was on the number of elliptic arcs and of the line segments on the numerical range's boundary. It was proved here that if $n \geq 4$ then the biggest number of line segments on $\partial W(A)$ is $2n-2$ while for $n \geq 3$, the biggest number of arcs of an ellipse $\partial W(A)$ can have is at most $n-2$ in addition to the sharpness of both upper bounds. The proof of this theorem relied on [70, Lemma 5] which relates the numerical ranges of submatrices of S with the line segments on $\partial W(S)$.

Tam [150] studied the numerical ranges of square complex matrices B in relation to block shift matrices and in [150, Theorem 1] found out that such numerical ranges are origin centered circular discs whenever the hermitian part of $e^{i\theta}B$ has the largest eigenvalue for all $\theta \in \mathbb{R}$. Clearly this result shows that a block-shift matrix being permutationally similar to a square complex matrix B is equivalent to $W(A)$ being an origin centered circular disc for some complex matrix A. Tam and Yang [151] have given further new equivalent conditions where they proved a result corresponding to the above and established that, under an origin centered rotation through an angle of $2\pi/k$, with k being an integer greater than or equal to 2, the numerical range of a complex matrix is invariant. Similarly the question on whether the numerical range of a matrix is an origin centered disk was reduced to the case proved in [151, Theorem 3] where the matrix being considered has a connected undirected graph.

Khalagai [91] discussed the numerical range's topological properties and showed how such properties as self-adjointness, uniticity, similarity and normality for bounded operators can be achieved through imposing certain conditions on their numerical ranges. Additionally normality was shown to be attained through the numerical range for classes of operators such as quasinormal, paranormal and hyponormal. Similarly, he showed that positivity of a product of operators can be achieved through the use of numerical range.

Tso et al [157] has shown that if T is a quadratic operator on a Hilbert space, then (1) the numerical range of T is an (open or closed) elliptical disc (or its degenerate form) and (2) for every $n \geq 1$, the n^{th} matricial range of T consists of n x n matrices whose numerical ranges are contained in the closure of the numerical range of T.

For a continuous linear operator S on a complex Hilbert space X, Embry [62, Theorem 1] letting $M_{z_o} = \{y : \langle Ay, y\rangle = z_o\|y\|^2\}$ for each $z_o \in \mathbb{C}$ and γM_{z_o} being M_z's linear span with $M_{z_o} \oplus M_{z_o} = \{y + t : y, t \in M_{z_o}\}$, characterized each $z_o \in W(S)$ in terms of M_{z_o} and established that z_o is either an interior point, or an extreme point or a non-extreme boundary point. The theorem relies on Embry [62, Lemma 1 and Lemma 2] for its proof. Modifying slightly the argument as given by Halmos [74, pp.317] on the proof of convexity of the numerical range of A shows that $sx - (1 - s)\lambda y \in M_z$ for some real $s \in (0, 1)$ and together with the homogeneity of M_z aids in proving the last assertion of [62, Theorem 1].

While studying the numerical range, Sims [146] considered the algebra and spatial numerical ranges of an operator on an element of a unital Banach algebra and on a normed linear space respectively as established by Bonsall and Duncan [34] and Lumer [110]. It is also in this study where an additional result is proved on approximating the convex hull of an element's spectrum by the numerical range of such an element defined in terms of the algebra's equivalent renormings which gives sharper versions of Williams' [160] result. For spatial numerical range, closure properties were also studied and a weaker result, compared to Berberian's, was obtained for Hilbert spaces. Similarly, it was also shown that for a compact operator on a l_p space or a Hilbert space, its spatial numerical range contain all extreme points of its closure which are non-zero implying that it contains the origin if and only if it is closed.

Following Williams's result [161] that gave a condition for the closure of the numerical range to contain zero, Furuta and Nakamoto [67, Theorem 2] gave an alternative to this result in which the numerical radius is replaced by the norm. The proof of this theorem imitates that of Williams in all except for the interchange between norm and numerical radius.

Williams [161] has also shown that being convexoid is equivalent to being normaloid for an operator. This is what Mecheri [114] sought to prove that the part "if" in that work is not true by giving an example which contradicts that part. This example is also the result by Mecheri [114, Proposition 2.7] which also proves the "only if" part of the williams' result and characterizes the numerical range by the convexoid operator.

Considering a normal operator, Ching-Kwong and Yiu-Tung [43] characterized the boundary points of $W(T)$ which together with *int* $(co(\sigma T))$ form $W(T)$. The result [43, Theorem 2] shows that the points give a direct sum decomposition of S which otherwise could not be identified using $W(S)$'s geometrical features. This result states as follows:

Theorem 2.8. *([43], Theorem 2) Suppose $S \in B(H)$ is normal. Then H allows a decomposition $H_o \oplus H_1$ which is orthogonal if and only if $\alpha \in W(S)$ is a boundary point such that $S = S_1 \oplus S_2 \in B(H_o \oplus H_1)$ where $\alpha \in W(S) \subseteq L_o$ with L_o a straight line and $W(S_2) \cap L_o = \emptyset$.*

In Theorem 2.8, $W(S_1)$ is either a line segment or a point which contains all of its endpoints, or one of its endpoints or none and $W(S_2)$ is a closed set or an open set or none of the two. An example was given which together with the theorem give an analysis of an operator whose numerical range is contained on a straight line in the complex plane.

Halmos [75, Problem 168] had raised the question of determining infinite-dimensional Hilbert space linear operators S with closed numerical range. An operator S defined on an l^2 space by $S(\alpha_1, \alpha_2, ..., \alpha_n) = (\alpha_1, \frac{1}{2}\alpha_2, ..., \frac{1}{n}\alpha_n))$ was used as an example where $W(S) = (0, 1]$. This example shows that even for compact operators, the numerical range might not be closed [75, Solution 168], a thing which motivated Sims [55] to construct the result [55, Theorem 1] contributing further knowledge on the numerical range closure properties for compact operators.

2.4 Spectra of Operators

In a finite dimensional space the spectrum of an operator is made up of its eigenvalues. A lot has been written about the spectrum of various operators by various authors. Among these authors is Shifang et al [144] who, for an upper triangular operator matrix of order 2 x 2, completely characterized the residual spectrum, the continuous spectrum and the point spectrum of this operator acting on the sum of Hilbert spaces $K \oplus H$ or Banach spaces $Y \oplus X$. Among the results which had a big impact is the following:

Theorem 2.9. *([144], Theorem 4) With the pair $(C, D) \in B(Y) \times B(X)$, we* *get*

$$((\rho(C) \cap \sigma_c(D)) \cup (\sigma_c(C) \cap \rho(D)) \cup \sigma_c(C) \cap \sigma_c(D)) \quad = \bigcap_{A \in B(X,Y)} \sigma_c(V_A)$$

with $\rho(C) = \mathbb{C} \setminus \sigma(C)$ *and* $\rho(D) = \mathbb{C} \setminus \sigma(D)$

It is the set $\bigcap_{C \in B(X,Y)} \sigma_c(V_A)$ which this theorem characterizes. To prove this theorem, [144, Lemma 1, Lemma 2, Lemma 3] are cited as useful lemmas.

Halmos [76] established that operators which are similar will always have the same compression, approximate point and point spectra. Atkinson et al [18] worked and proved the non-emptiness of the essential spectrum of an operator whose resolvent is compact. Akkouchi [7] while looking for a way to characterize λ in the spectrum of a normal operator, introduced the T_λ-spectral sequence ($T_\lambda = T - \lambda I$), a concept used in the discussion of its nature and in classifying normal and bounded operators' spectral points with respect to properties of spectral sequences associated with them. All this is done in [7, Theorem 2] which is mainly about characterizing a bounded operator on a Hilbert space's point spectrum. It uses extensively the concept of T_λ in its proof. Closely related to the above, [7, Theorem 4] is another result which confirms the non-emptiness of an operator's point spectrum.

Gustafson and Rao [73] have shown that a normal operator's numerical range is its spectrum's convex hull and its closure contains its spectrum. Laura [103] studied bounded linear operators on a Banach space Y and considered their Banach algebra's element S looking at its spectrum's continuity and its relation to that of its adjoint S^*. He showed that the spectrum function is likely to be discontinuous at A^* but be continuous at A. Hladnik et al [80] showed that two operators on a Hilbert space have a product with real spectrum if one of them is symmetric and the other is positive. It is further shown that the product of two operators has positive spectrum. Klein [97] considered a unit circle's Hardy space H^2 and computed explicitly the numerical range of a Toeplitz operator on this space establishing that it is only the given Toeplitz operator's spectrum which influences the numerical range of this operator.

Sims [145, Theorem 1] took a bounded linear operator A on a complex Hilbert space and established that if $\lambda \in W(A)$ is $\overline{W(A)}$'s corner vertex, then $\lambda \in \rho\sigma(A)$. This related points in $\sigma(A)$ to $\overline{W(A)}$'s corner vertices.

Concerned about the size of the numerical range of an operator similar to $S \in B(H)$, Williams [162, Theorem 2] generalized the classical result by Akemann and Anderson [6] which characterized matrices S of order $n \times n$ such that the right half plane $H^+ = \{z : Rez > 0\}$ contain their spectra. In the theorem an operator S is considered together with an open convex set C which contain $\sigma(S)$. It was then established that an operator S_o which is invertible exists such that $C \supset \overline{W(S_o^{-1}AS_o)}$. This is connected to the result [162, Theorem 4] which asserts that for an operator S, $\sigma(S) \subset H^+$ if and only if there exist invertible operators $S_o, S_1 \in B(H)$ such that $\overline{W(S_o^{-1}SS_o)} \subset H^+$, $\overline{W(S_1^{-1}SS_1)} \subset H^+$ and $\overline{W(SS_1)} \subset H^+$ as long as S_1 is positive. This simply shows that there must exist an operator S_1 which is invertible and positive for $\sigma(S)$ to be in the positive half of the complex plane and the real part of SS_1 to be invertible and positive.

Similarly a finite-dimensional version of Ostrowskki and Scheneider [126] result for matrices of order $n \times n$ is [162, Theorem 6] which characterizes such operators as S whose spectra do not intersect the imaginary axis. The proof of this result is a consequence of [162, Theorem 4] giving a necessary and sufficient condition for S to have an invertible and positive $Re\ SM$ for a given self-adjoint and invertible M.

While discussing Toeplitz-Hausdorff theorem, Halmos [75] claims that the proof of the numerical range's convexity is only a computational process. This assertion influenced Douglas and Robin [58] who went a head to show that the classical proof of the theorem fails by using invalid intuitionistic logic principles and noted that due to the inability to be translated into the language of recursive function theory, the proof cannot be computational as claimed by Halmos. However, an effort was made to show the extent to which those failures could be addressed using the intuitionistic logic and as a consequence of this, results [58, Theorem 1 and Theorem 2] were proved to confirm the convexity of the numerical range. These results actually repaired Halmos' breakdown of the same result.

Since the closure of the numerical range of a bounded normal operator is known to be exactly its spectrum's convex hull [149], Ching-Hwa [42] named this as property A and that of the spectrum lying on a convex curve he called property B. This enabled the construction of the result [42, Theorem 1] which showed that having property A does not necessarily imply that an operator is normal and therefore more is needed to be known about an operator for it to be classified as normal or not. In [42, Theorem 2] it is established that S with properties A and B can be written as $S_1 \oplus S_2$ which is an operator defined on $S_1 \oplus S_2$, a product space having H_1 spanned by the characteristic elements of S_1 and H_2 spanned by those of S_2 such that:

(i) S_1 is normal, and $\sigma(S_1) = \overline{P\sigma(S_1)}$,

(ii) $\sigma(S_2) = C\sigma(S_2)$,

(iii) S being normal is equivalent to S_2 being normal.

Consequently this result implies that $\sigma_r(S) \subseteq \sigma_r(S_1) \cup \sigma_r(S_2)$ after defining an operator's spectrum on an empty subspace as the empty set. Generally, what the above results show is that S is not necessarily required to be normal and that not much can be said about an operator until more else is known. It is noted that unitary operators are a class of operators with this property.

Jordan Bell [89] proved that a bounded linear operator's spectrum is a nonempty subset of $\mathbb{C}$ and that it is only real if the operator is self-adjoint. In this work, proofs of various results about resolvents were used to show that the spectrum is nonempty. Among these results is that in [89, Theorem 5] which says that whenever $S \in \mathcal{B}(H)$ and $|\lambda| > \|S\|$ then λ is always in the resolvent of S. The result simply confirms that the spectrum of the operator S is always a bounded set contained in the disc $|\lambda| \leq \|S\|$.

Bӧttcher et al [37, Theorem 1.1] showed that given a bounded linear operator S defined on an infinite- dimensional Hilbert space which is separable and an orthogonal compression S_n of S to the span of the first n elements of an orthonormal basis of H, then the sequence $a_k(S_n)$ of approximation numbers converge to $a_k(S)$ as $n \to \infty$ for each $k \geq 1$. It is noted that the observation allows the identification of all lower and upper eigenvalue limits when S is self-adjoint. The points of a self-adjoint operator's spectrum lying outside the essential spectrum's convex hull are given by these limits. The result states as follows:

Theorem 2.10. *([37], Theorem 1.1) If* $A \in B(H)$ *then* $\lim_{n\to\infty} s_k(A_n) = s_k(A)$*, for each* $k \geq 1$.

The implication of this result is that a self-adjoint operator S's spectrum having a connected essential spectrum can completely be retrieved from eigenvalues of S_n as $n \to \infty$.

Following up on the ideas by Arveson [15] on the approximation of linear operators' spectra , Hansen [77] clearly presented methods of approximating spectra of various types of linear operators on H. Here known methods used for finite-dimensional matrices have been taken and generalized to infinite-dimensional set up and used to approximate spectra in various classes of operators. In the end a proposition is generated which solves the general approximation problem for the spectra of arbitrary bounded operators when the n-pseudospectrum is introduced with arguments made on how this can be taken to be the spectrum's approximation. A keen observation reveals that the result is equivalent to Theorems 2.3 and 3.8 in [14] and similar approaches are used for its proof. But as the finite section technique and the conquer-and-divide approach are not similar, the theorems cannot be used directly. However much stronger estimates are given by this result on how the false eigenvalues may behave in case they occur.

2.5 Norms of Operators

Khalid and Albadawi [93] used techniques such as the Minkowski and Cauchy-Schwarz inequalities to come up with sharp norm inequalities for operators' absolute values. Timoney [152] considered elementary operators on $B(H)$ and established their completely bounded norms in addition to the lower bounds for the norms, hence proving Mathieu's conjecture [112] which had sought to establish the validity of $\|T_{a.b}\| \geq c\|a\|\|b\|$ holding in general where $c = 1$. As special cases, results for $\|T_{a.b}\|_{cb}$ and $\|T_{a.b}\|$ were also established.

For a positive linear operator $0 \leq S : L^q(X, \mu) \rightarrow L^p(Y, \nu)$ and $\|S\|_{q,p}$ denoting its operator norm, Ralph and Anton [133] gave a method of computing the exact value of $\|S\|_{q,p}$ or of bounding it from above. Applying this for the voltera operator $Vf(t)dt$, its exact norm $\|V\|_{q,p}$ was computed.

Further, Timoney [154] studied elementary operators in the context of operator algebras on Banach spaces and C*-algebras where a lack of symmetry in the norm problem was established on Banach spaces containing the finite rank operators and also for the calkin algebra but at the same time showed an isormophic symmetry for sub-homogeneous C^*-algebras. Okelo et al [124] studied the Jordan Elementary operator $U_{A,B(X)} = AXB + BXA, \forall X \in B(H)$ where A, B are fixed in $B(H)$ and determined its norm. Particularly, they proved that $\|U_{A,B}\| \geq \|A\|\|B\|$. The work considered a Jordan elementary operator in C*-algebras .

Timoney [153] proved that on the completely bounded norm of elementary operators, the Haagreup estimate is the best for $B(H)$ through a generalization of Stampfli's theorem. It was shown that if S is an elementary operator whose length is ℓ, then for $k = \ell$, its completely bounded norm is equal to the $k-$norm. For a C^*-algebra, this is true if and only if the algebra is $k-$subhomogeneous or an antiliminal C^*-algebra restricted to a $k-$subhomogeneous algebra.

Osikiewicz and Tonge [125] looked at Hardy-Littlewood estimates for norms of operators on sequence spaces and used recent advances in interpolation theory to provide relatively simple proofs. Equivalence of various norms on the unitization of a nonunital Banach algebra were established by Gaur and Kovarik [71] with uniform bounds over the class of such algebras. Here, a tighter bound, 3, was obtained in C*-algebras for elements with Hermitian non-unital parts where the following theorem was established.

Theorem 2.11. *([71], Theorem 1).For every nonunital Banach algebra S with unitization S^+ and with regular norm, and for every $\lambda \in \mathbb{C}$ and $x \in S$, we have*

$$\|\lambda e + x\| \leq \|\lambda e + x\|_1 \leq (6 \exp 1)\|\lambda e + x\|.$$

If S is a C-algebra with hermitian $x \in S$ and complex number λ, then $\|\lambda e + x\|_1 \leq 3\|\lambda e + x\|_{op}$ and 3 is best possible minimal.*

Regular norms of C*-algebras with Hermitian elements have been considered and uniform equivalence of the two unitization norms over the class of nonunital Banach algebras with regular norms established.

Berg [25] answered the interesting problem of determining whether for each operator T on a separable Hilbert space and each polynomial $\mathcal{P}$ there exists a compact operator K such that the norm of $\mathcal{P}(T + K)$ is the essential norm of $\mathcal{P}(T)$ by showing that if there is a continuous deformation of T to 0 such that $\mathcal{P}(T)$ shrinks properly in essential norm to 0, then there does indeed exist such a compact operator K.

Lancaster [101] considered the construction of a norm on a direct sum of normed linear spaces and called a norm absolute if it depends only on the norm of the component spaces. Several characterizations were given of absolute norms. Absolute norms were then used to construct norms on tensor products of normed linear spaces and on tensor products of operators on normed linear spaces. Okelo [123] studied the structural properties of elementary operators where Dvoretsky's theorem and its applications were utilized to establish the norm of a symmetrized two-sided multiplication operator acting on a C*-algebra. Results on the norms of matricial operators were also included in this study.

Lin [107] presented various types of characterizations of a bounded linear operator T on a Hilbert space whose norm is an eigenvalue for T and gave their consequences. It was established that many results in Hilbert space operator theory are related to such an operator. This study was motivated by results from two works namely:

(i). The findings of Abramovich et al [1] through the following theorem:

   Theorem 2.12. *([1], Theorem 2.7) A linear compact operator on a locally uniformly convex Banach space into itself satisfies the Daugavet equation if and only its norm is an eigenvalue for the operator*

(ii). The result of Halmos [76] that every compact operator on a Hilbert space has a norm-attaining vector for the operator.

While noting that the problem of characterizing all Supremum norms on a space of matrices or linear transformations was still unsolved, Scheneider and Strang [138] came up with the following result:

Theorem 2.13. *([138], Theorem 3) Let V and U be finite-dimensional vector spaces and let $\nu, \mu, i = 1, 2$ be norms on V and U respectively, and let m_1, n_2 be defined by $m_1 = \inf_V \frac{\nu_2}{\nu_1}, n_2 = \inf_V \frac{\mu_1}{\mu_2}$ be non-zero. Let S be a set of bounded transformations which contains all bounded linear transformations of rank 1. Let $\sup_1$ and $\sup_2$ be the norms on S belonging to ν, μ and ν_2, μ_2. Then $\inf_S \frac{\sup_1}{\sup_2} = m_1 n_2$.*

This result together with others was intended as a step towards solving the problem. It was the most general in which the assumption of finite dimensionality is not needed.

Shebrawi and Albadawi [143] proved that if $A_i, B_i, X_i (i = 1, 2, ..., n)$ are operators in $B(H)$ such that X_i is a self-adjoint operator and $0 < r \leq 1$, then

$$|||\sum_{i=1}^{n}|A_i^* X_i B_i + B_i^* X_i A_i|^r||| \quad \leq \quad 2n^{1-\frac{r}{2}}\sum_{i=1}^{n}|||(A_i^*|X_i|A_i)^r|||^{\frac{1}{2}}|||(B_i^*|X_i|B_i)^r|||^{\frac{1}{2}}$$

leading to the following inequality:

$$|||\mid A\mid^{2r} - \mid B\mid^{2r}||| \quad \leq \quad 2^{1-r}|||\mid A+B\mid^{2r}|||^{\frac{1}{2}}|||\mid A-B\mid^{2r}|||^{\frac{1}{2}} \qquad (2.5.1)$$

Inequality 2.5.1 is a generalization of the result given by Bhatia [27] as follows:

$$|||\mid A-B\mid||| \quad \leq \quad \sqrt{2}|||\mid A+B\mid|||^{\frac{1}{2}}|||\mid A-B\mid|||^{\frac{1}{2}}.$$

A generalization of some unitarily invariant norm inequalities for some absolute values have been given in the following two results by Ali, Yang and Shakoor [11].

Lemma 2.14. *([11], Lemma 2.1) Let* $A_i, B_i \in B(H), i = 1, 2, ..., n$*. Then*

$$(2n)^{\frac{1}{2}-\frac{1}{r}}|||\sum_{i=1}^{n}\mid A_i + B_i\mid^r|||^{\frac{1}{r}} \quad \leq \quad 2^{\frac{1}{2}-1}(|||\sum_{i=1}^{n}\mid A_i\mid^r|||^{\frac{1}{r}} + |||\sum_{i=1}^{n}\mid B_i\mid^r|||^{\frac{1}{r}}).$$

for $0 < r \leq 1$,

$$n^{-|\frac{1}{r}-\frac{1}{2}|}|||\sum_{i=1}^{n}\mid A_i + B_i\mid^r|||^{\frac{1}{r}} \quad \leq \quad |||\sum_{i=1}^{n}\mid A_i^r\mid|||^{\frac{1}{r}} + |||\sum_{i=1}^{n}\mid B_i\mid^r|||^{\frac{1}{r}}$$

for $r \geq 1$, *and*

$$n^{\frac{-(1-\frac{1}{p})}{r}} ||| \sum_{i=1}^{n} \mid A_i + B_i \mid^r |||^{\frac{1}{p}}_{\frac{r}{}} \quad \leq \quad ||| \sum_{i=1}^{n} \mid B_i \mid^r |||^{\frac{1}{r}}_{p}.$$

for $1 \leq p, r < \infty$.

Lemma 2.15. *([11], Lemma 2.2) For* $A, B, X \in B(H)$*, for all unitarily invariant norms and for all positive real numbers* μ_1, μ_2 *and* r *such that* $\mu_1^{-1} + \mu_2^{-1} = 1$*, we have* $||| \mid A^*XB \mid^r ||| \leq ||| \mid AA^*X \mid^{\frac{\mu_1 r}{2}} ||| \, ||| \mid XBB^* \mid^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}$*, and also, if* f *and* g *are nonnegative continuous functions on* $[0, \infty]$ *satisfying* $f(t)g(t) = 1$*, for all* $t \in [0, \infty]$*, then we have*

$$||| \mid A^*XB \mid^r ||| \quad \leq \quad |||(A^*f^2(\mid X^* \mid)A)^{\frac{\mu_1 r}{2}}|||^{\frac{1}{\mu_1}} |||(B^*g^2(\mid X \mid)B)^{\frac{\mu_2 r}{2}}|||^{\frac{1}{\mu_2}}.$$

The two, among other several Lemmas, contain norm inequalities of Minkowski type and generalized form of Cauchy-Schwarz inequality.

In the study of what they called absolute norms and monotonic norms , Bauer et al [21] defined norm(x) to be a norm in n-dimensional complex coordinate space and $|x|$ to be the moduli of the components of the vector x. Norm(x) was then called monotonic if $|x| \leq |y|$ implies norm $x \leq$ norm y and a norm was called absolute if it depends only on the moduli of its components, that is if norm $x =$ norm$|x|$ for all x. Using the observation that the least upper bound of a diagonal matrix is the maximum of the moduli of the diagonal elements the class of these least upper bound norms and that of vector norms to which they are subordinate were characterized and some properties of these norms were shown. The following result was proved:

Theorem 2.16. *([21], Theorem 2) An absolute norm is monotonic and vice* *versa.*

This result actually shows that a monotonic norm is always strictly homogeneous.

In answering the question: For which pairs of C^*-algebras A and B are the Haagerup norm($\|.\|_h$), operator space projective norm($\|.\|\Lambda$) and Banach space projective norm($\|.\|\gamma$) equivalent? Itoh [84] claimed that the Haagerup norm $\|.\|_h$ and the Banach space projective norm $\|.\|_\gamma$ are equivalent on the algebraic tensor product $A \otimes B$ of two C^*-algebras A and B if and only if A or B is subhomogeneous. Kumar and Sinclair [98] disputed this and asserted that this claim is false as can be seen by considering the norms on $l^\infty \otimes B(H)$ or $K^\infty \otimes K(H)$. The authors gave their correct version in [98, Theorem 6.1]. It should be noted that if the Banach space projective norm is replaced by the operator space projective norm the same conclusion will hold as indicated in [98, Theorem 7.4] In general the results above confirm that the Haagerup norm on the tensor product $S_1 \otimes S_2$ of two C^*-algebras S_1 and S_2 are Banach space equivalent to either the Banach space projective norm or the operator space projective norm if and only if either S_1 or S_2 is finite dimensional or S_1 and S_2 are infinite dimensional and subhomogeneous. At the same time, it is shown that the operator space projective norm and the Banach space projective norm are equivalent on $S_1 \otimes S_2$ if and only if S_1 or S_2 is subhomogeneous.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

Methods and techniques useful in obtaining results in the next chapter are given here. The methodology involved the use of well known inequalities and equalities such as the Cauchy-Schwarz inquality, the triangle inequality, the parallelogram law and the polarization identity. Diagonalization process was used to find the eigenvalues of matrix operators which give the spectra of such operators. We also restated and described some known useful results without their proofs.

3.2 Inequalities and Equalities

3.2.1 Cauchy-Schwarz inequality

Given $\zeta, \varsigma \in X$ where X is an inner product space we have $|\langle \zeta, \varsigma \rangle| \leq \|\zeta\| \|\varsigma\|$ [166].

Indeed, when $\varsigma = 0$, it is trivial that $\langle \zeta, \varsigma \rangle = 0$. Let $\varsigma \neq 0$ and $\mu \in \mathbb{C}$, then

$$0 \leq \|\zeta + \mu\varsigma\|^2 = \|\zeta\|^2 + \overline{\mu}\langle \zeta, \varsigma \rangle + \mu\langle \varsigma, \zeta \rangle + |\mu|^2\|\varsigma\|^2. \tag{3.2.1}$$

Putting $\mu = -\langle \zeta, \varsigma \rangle \|\varsigma\|^{-2}$ in equation 3.2.1 we have

$$\begin{aligned} 0 &\leq \|\zeta\|^2 - |\langle \zeta, \varsigma \rangle|^2\|\varsigma\|^{-2} - |\langle \zeta, \varsigma \rangle|^2\|\varsigma\|^{-2} + |\langle \zeta, \varsigma \rangle|^2\|\varsigma\|^{-4}\|\varsigma\|^2 \\ &= \|\zeta\|^2 - |\langle \zeta, \varsigma \rangle|^2\|\varsigma\|^{-2}. \end{aligned}$$

Implying that

$$\begin{aligned} 0 &\leq \|\zeta\|^2\|\varsigma\|^2 - |\langle \zeta, \varsigma \rangle|^2 \\ |\langle \zeta, \varsigma \rangle|^2 &\leq \|\zeta\|^2\|\varsigma\|^2 \\ |\langle \zeta, \varsigma \rangle| &\leq \|\zeta\|\|\varsigma\|. \end{aligned}$$

3.2.2 Triangle inequality

Let $\omega, \zeta \in \mathbb{R}^n$. Then we have [12], $\|\omega + \zeta\| \leq \|\omega\| + \|\zeta\|$.
Indeed, $\|\omega + \zeta\|^2 = \langle \omega + \zeta, \omega + \zeta \rangle = \langle \omega, \omega \rangle + \langle \omega, \zeta \rangle + \langle \zeta, \omega \rangle + \langle \zeta, \zeta \rangle$
Therefore by cauchy-schwarz inequality we have,
$\|\omega + \zeta\|^2 \leq \|\omega\|^2 + 2\|\omega\|\|\zeta\|\| + \|\zeta\|^2$ which gives
$\|\omega + \zeta\| \leq \|\omega\| + \|\zeta\|$.

3.2.3 Schwarz-inequality for positive operators

If S is a positive operator in $B(H)$ [96], then $|\langle S\varrho, \varsigma \rangle|^2 \leq \langle S\varrho, \varrho \rangle \langle S\varsigma, \varsigma \rangle \ \forall \varrho, \varsigma \in H$.

3.2.4 Parallelogram law

Given an inner product space X [105], then $\|x+z\|^2+\|x-z\|^2 = 2(\|x\|^2+\|z\|^2)$ holds for all $x, z \in X$ where $\|x\| = \sqrt{\langle x, x\rangle}$ and $\|z\| = \sqrt{\langle z, z\rangle}$.

3.2.5 Polarization identity

Let X be an inner product space [105]. Then for $\zeta, z \in X$ we have $\langle \zeta, z\rangle = \frac{1}{4}\{\|\zeta + z\|^2 - \|\zeta - z\|^2 + i\|\zeta + iz\|^2 - i\|\zeta - iz\|^2\}$.

3.2.6 Bessel's inequality

Suppose [118] $\{e_1, e_2, e_3, , ..., e_n\}$ is a finite orthonormal set in H. Then for any element $g \in H$ we have $\sum\limits_{i=1}^{n} |\langle g, e_i\rangle|^2 \leq \|g\|^2$ and $g - \sum\limits_{i=1}^{n} \langle g, e_i\rangle e_i \perp e_j$ $(j = 1, 2, ..., n)$.

3.3 Fundamental Theorems

Theorem 3.1. *(Riesz Representation Theorem) [49]: Suppose we have a linear functional ϕ which is bounded on H. Then*

(i). There exists a vector $\xi \in H$ which is unique giving $\phi(\xi) = \langle \xi, \zeta\rangle$ for each $\zeta \in H$

(ii). $\|\phi\| = \|\zeta\|$.

Theorem 3.2. *(Toeplitz-Hausdorff Theorem) [141]: Let $S \in B(H)$. Then the numerical range of S is convex.*

3.4 Useful Results

Corollary 3.3. *Let $S \in NA(H)$.*

Then $\gamma(S) = \inf\{\|TAT^{-1}\| : T$ is invertible on $H\}$

Proof. See [141, Corollary 6.2] □

Proposition 3.4. *Let V be a normed vector space, W be a proper linear subspace of V which is closed and $v_0 \in V \backslash W$. Then we have $v^* \in V^*$ to give $v^*(w) = 0$ for all $w \in W$, $v^*(v_0) = d(v_0, W) = \inf_{w \in W} \|v_0 - w\| > 0$ and $\|v^*\| = 1$.*

Proof. See [139, Proposition 4.19] □

Corollary 3.5. *Given a normed vector space A and a linear subspace B of A, $a^* \in A^* \backslash \{0\}$ is equivalent to B not being dense in A $\forall b \in B$.*

Proof. See [139, Corollary 4.23] □

Proposition 3.6. *Let $S \in NA(H)$ be a two by two- cross diagonal matrix operator. Then $W(S)$ is either an elliptical disc with foci at the eigenvalues or a line segment joining the eigenvalues.*

Proof. See [141, Proposition 3.1] □

Theorem 3.7. *Let S_1 and S_2 be operators on H. Then if $0 \notin W(S_1)$, we have $\sigma(S_1^{-1}S_2) \subset \overline{W(S_1)}/\overline{W(S_2)}$.*

Proof. See [163, Theorem 1] □

Theorem 3.8. *Let $S \in B(H)$. Then $\sigma(S) \subset \overline{W(S)}$.*

Proof. See [73, Theorem 1.2-1] □

3.5 Technical approaches

3.5.1 Diagonalization

If a matrix has non-zero entries only in its main diagonal with zeroes everywhere else then it is said to be diagonal [102]. Let Q be an $n \times n$ matrix with complex entries. Then Q is diagonalizable whenever there exist complex matrices P, D each of order $n \times n$ such that D is diagonal given by $D = PQP^{-1}$.

Example 3.9. Let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$. Then characteristic equation is

$$\lambda^2 - trac(A)\lambda + detA = 0$$

i.e

$$\lambda^2 - 3\lambda + 2 = 0$$

which gives

$$(\lambda - 2)(\lambda - 1) = 0$$

so that

$$\lambda_1 = 1, \lambda_2 = 2.$$

Now, when $\lambda_1 = 1$ we have null $A - \lambda I$ given as $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$. Hence,

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Similarly when $\lambda = 2$, we get

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, $P = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.
This implies that

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

Here, numerical ranges and spectra of norm-attaining operators in C^*-algebras are characterized. Norms of norm-attaining operators in C^*-algebras are also established via the lower bound and upper bound estimates.

4.2 Numerical ranges in $NA(H)$

In this section, numerical range results are given for norm-attaining operators. We begin with a proposition on elements of the numerical range. This is a known result in Banach algebra in general. However we restate it and give an analogous proof in the context of the C*-algebra of norm-attaining operators.

Proposition 4.1. *Let $S_o \in NA(H)$. Then $W(S_o)$ is non-empty and $0 \in W(S_o)$ if S_o is compact.*

Proof. Suppose S_o is compact. Then there exists a sequence $\{\varsigma_n\} \in H$ of unit vectors and $\varsigma \in H$ with $\|\varsigma\| = 1$ for which $\{S_o\varsigma_n\} \to \varsigma$ and $\|S_o\varsigma\| = \|S_o\|$. Let

$\lambda \in W(S_o)$, then $\lambda = \langle S_o\varsigma_n, \varsigma_n\rangle$. This gives

$$\begin{aligned} |\lambda| &= |\langle S_o\varsigma_n, \varsigma_n\rangle| \\ &\leq \|S_o\varsigma_n\|.\|\varsigma_n\| \\ &\leq \|S_o\|.\|\varsigma_n\|.\|\varsigma_n\| \\ &\leq \|S_o\| \end{aligned}$$

implying that $\lim_{n\to\infty}\langle S_o\varsigma_n, \varsigma_n\rangle = \|S_o\|$ and therefore $\{\langle S_o\varsigma_n, \varsigma_n\rangle\}_{n=1}^{\infty}$ is bounded. Now,

$$\begin{aligned} \langle S_o\varsigma_n - \|S_o\|\varsigma_n, S_o\varsigma_n - \|S_o\|\varsigma_n\rangle &= \langle S_o\varsigma_n, S_o\varsigma_n\rangle - \langle S_o\varsigma_n, \|S_o\|\varsigma_n\rangle \\ &- \langle \|S_o\|\varsigma_n, S_o\varsigma_n\rangle + \langle \|S_o\|\varsigma_n, \|S_o\|\varsigma_n\rangle \\ &= \|S_o\varsigma_n\|^2 - \|S_o\|(\langle S_o\varsigma_n, \varsigma_n\rangle + \langle\overline{S_o\varsigma_n, \varsigma_n}\rangle) + \|S_o\|^2\|\varsigma_n\|^2 \\ &\leq \|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\||\langle S_o\varsigma_n, \varsigma_n\rangle| + \|S_o\|^2\|\varsigma_n\|^2 \\ &= 2\|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\|\langle S_o\varsigma_n, \varsigma_n\rangle \\ &= 2\|S_o\|^2\|\varsigma_n\|^2 - 2\|S_o\|^2\|\varsigma_n\|^2 \\ &= 0. \end{aligned}$$

This means $\{\varsigma_n\}_{n=1}^{\infty} \to 0$ weakly in H and therefore $\|S_o\|$ is an eigenvalue of S_o. Since each eigenvalue of S_o is contained in $W(S_o)$ then $W(S_o)$ is nonempty. Let $S_o = S_o^*$ (since by Hahn Banach Theorem, if S_o attains the norm S_o^* also does) and particularly let $\lambda = 0$ and ς be a unit vector with the aim that $S_o\varsigma = 0$ but $S_o^*\varsigma \neq 0$. Let $f = \frac{S_o^*\varsigma}{\|S_o^*\varsigma\|}$. As $\langle\varsigma, S_o^*\varsigma\rangle = \langle S_o\varsigma, \varsigma\rangle = \langle 0, \varsigma\rangle = 0$, it means ς, f is orthonormal and hence spans any subspace M which is two-dimensional. This therefore is an implication that the numerical range of S_o when restricted to M

is contained in $W(S_o)$ that is, $W(S_o|_M) \subset W(S_o)$. Therefore showing that 0 is in int$(W(S_o|_M))$ is all what is needed in this case. Using orthonormal basis ς, f for M, $\begin{pmatrix} 0 & c \\ 0 & * \end{pmatrix}$ is the way the matrix of $S_o|_M$ appears where $c = \langle S_o|_M f, \varsigma\rangle$. It should be shown now that $c \neq 0$ so that $W(S_o|_M)$ is confirmed not to be a degenerate disk which is elliptical in form with a focus found at the origin. We have, $c = \langle S_o|_M f, \varsigma\rangle = \langle S_o f, \varsigma\rangle = \langle f, S_o^*\varsigma\rangle$. But $f = \frac{S_o^*\varsigma}{\|S_o^*\varsigma\|}$ which implies that $\frac{\langle S_o^*\varsigma, S_o^*\varsigma\rangle}{\|S_o^*\varsigma\|} = \|S_o^*\varsigma\| \neq 0$ as required. □

The numerical range of a norm-attaining operator is related to that of its adjoint as shown in the result which follow:

Proposition 4.2. *Let $S_o \in NA(H)$. Then $W(S_0^*) = \{\overline{\beta} : \beta \in W(S_o)\}$.*

Proof. Suppose $\beta \in W(S_o^*)$. Then a unit vector $\xi \in H$ exists so that $\beta = \langle S_o^*\xi, \xi\rangle = \langle \xi, S_o\xi\rangle$ which implies that $\overline{\beta} = \overline{\langle \xi, S_o\xi\rangle} = \langle S_o\xi, \xi\rangle$ and therefore $\overline{\beta} \in W(S_o)$. But as $\overline{\overline{\beta}} = \beta$, then it is true that $W(S_o^*) \subseteq \{\overline{\beta} : \beta \in W(S_o)\}$. Similarly, let $\alpha \in \{\overline{\beta} : \beta \in W(S_o)\}$. Then this implies that $\beta \in W(S_o)$. Therefore, $\exists$ some vector $\xi \in H$ where $\|\xi\| = 1$ so that $\alpha = \langle S_o\xi, \xi\rangle$ implying that $\overline{\alpha} = \langle \xi, S_o^*\xi\rangle$. This means that $\alpha = \overline{\overline{\alpha}} = \overline{\langle \xi, S_o^*\xi\rangle} = \langle S_o^*\xi, \xi\rangle$. Therefore we have $\{\overline{\beta} : \beta \in W(S_o)\} \subseteq W(S_o^*)$. Hence, $W(S_o^*) = \{\overline{\beta} : \beta \in W(S_o)\}$. □

Proposition 4.3. *Suppose $S \in NA(H)$ and $[NA(H)]_1$ is the open unit disc of $NA(H)$ with center at the origin . Then $W(S) = [NA(H)]_1$ if S is a backward unilateral shift.*

Proof. Since $[NA(H)]_1$ is open, then for some $\lambda \in \mathbb{C}$, $\lambda \in [NA(H)]_1$ implies that $|\lambda| < 1$. As S is norm-attaining, there must be a vector $\xi \in H$ with $\|\xi\| = 1$ so that $\|S\xi\| = \|S\|$. If λ is in $W(S)$, then $|\lambda| = |\langle S\xi, \xi\rangle| \leq \|S\xi\|\|\xi\| \leq 1$ with equality satisfied whenever $S\xi$ and ξ are expressible as multiples of each other where $\|S\xi\| = 1$. Therefore this means that $S\xi = \lambda\xi$ where $|\lambda| = 1$. Let u_n be an orthonormal basis. Then $Su_1 = 0$ and $Su_n = u_{n+1}$ $\forall n \geq 1$. If $\xi = \sum\limits_{n\geq 1} c_n u_n$, then $S\xi = \lambda\xi$ implies that $\lambda c_n = c_{n+1}$ $\forall n \in \mathbb{N}$ which is impossible since $1 = \|\xi\|^2 = \sum\limits_{n\geq 1} |c_n|^2$. Hence $W(S) \subset [NA(H)]_1$. Now let $\lambda \in [NA(H)]_1$. This implies that $|\lambda| < 1$ and letting $\xi_0 = \sum\limits_{n\geq 1} \lambda^n u_n \in H$ means $S\xi_0 = \lambda\xi_0$ and therefore λ is an eigenvalue for S. Since $W(S)$ contains all the eigenvalues of S, then $\lambda \in W(S)$ which means $[NA(H)]_1$ is a subset of $W(S)$ implying that $W(S) = [NA(H)]_1$. □

The relationship between the convex and infinity convex hulls of a set of complex numbers is shown in the proposition which follows.

Lemma 4.4. *Let $\alpha = \beta_n$ be any set of numbers which is countable. If $\beta_n \in \mathbb{C}$, then $co_\infty(\alpha) = co(\alpha)$.*

Proof. Since $co(\alpha) \subset co_\infty(\alpha)$ with $co_\infty(\alpha)$ being convex, then we have to show that any $q \in co_\infty(\alpha)$ is a convex combination of α's points. Since

$$co(k\alpha + d) \quad = \quad k\ co(\alpha) + d, \forall k, d \in \mathbb{C} \tag{4.2.1}$$

then it is the same for $co_\infty(\alpha)$ and therefore α can be replaced by $\alpha - q$ taking q to be equal to 0.

Assume $0 \notin co(\alpha)$. Then there is a line between 0 and $co(\alpha)$ and therefore α, $co(\alpha)$ and $co_\infty(\alpha)$ are assumed to belong to the closed upper half of the complex-plane through using Equation 4.2.1 to have a rotation about the origin.

It is assumed that there exist infinitely many non-zero numbers k_n between 0 and 1 such that $0 = \sum\limits_{n=0}^{\infty} k_n\beta_n$, otherwise trivially $0 \in co(\alpha)$. Now, $0 =$

$\sum_{n=0}^{\infty} k_n Im(\beta_n)$ and since $Im\beta_n \geq 0$; $\forall n$, then there must be real β_n for every non-zero k_n. Hence there is some $\beta_n \in \mathbb{R} < 0$ and another $\beta_m \in \mathbb{R} > 0$. This therefore makes the origin to lie on this line between β_m and β_n and therefore is in α's convex hull. This is a contradiction since 0 was assumed not to belong to $co(\alpha)$. $\square$

The result which follow relates a norm-attaining unitarily diagonalizable operator's numerical range to its eigenvalues' convex hull.

Theorem 4.5. *Let $S \in NA(H)$ and $co(S)_{ev}$ be a convex hull of S's eigenvalues. Then $W(S) = co(S)_{ev}$ if S is unitarily diagonalizable.*

Proof. Let S be unitarily diagonalizable. Then we have a basis $\{e_j\}$ which is orthonormal for H together with a sequence $\{\lambda_j\}$ of complex numbers giving $Se_j = \lambda_j e_j$ for each integer $j \geq 0$. Hence

$$\begin{aligned}
W(S) &= \{\langle S\varsigma, \varsigma\rangle : \varsigma \in H, \|\varsigma\| = 1\} \\
&= \{\sum_{j=0}^{\infty} \langle S\langle \varsigma, e_j\rangle e_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \langle \varsigma, e_j\rangle \langle Se_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \langle \varsigma, e_j\rangle \langle \lambda_j e_j, \varsigma\rangle\} \\
&= \{\sum_{j=0}^{\infty} \lambda_j \langle \varsigma, e_j\rangle \overline{\langle \varsigma, e_j\rangle} \\
&= \{\sum_{j=0}^{\infty} \lambda_j |\langle \varsigma, e_j\rangle|^2 : \varsigma \in H, \|\varsigma\| = 1\} \\
&= \{\sum_{j=0}^{\infty} \lambda_j b_j : 0 \leq b_j \leq 1, \sum_{j=0}^{\infty} b_j = 1\} \\
&= co(S)_{ev}
\end{aligned}$$

where $(S)_{ev}$ is the collection of S's eigenvalues and this, together with Lemma

4.4, completes the proof. □

The Lemma which follows show that the foci of the ellipse which form a norm-attaining operator's numerical range represented by a matrix of order 2×2 are the matrix's eigenvalues.

Lemma 4.6. *Let $S \in NA(H)$ such that $S = \begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix}$, $q, r \in \mathbb{C}$. Then $W(S)$ is the ellipse whose center is the origin and foci $\pm(qr)^{\frac{1}{2}}$ having length of major axis as $|q| + |r|$ and $|\,|q| - |r|\,|$ being length of minor axis.*

Proof. As q and r are complex numbers, let us assume that they are both positive with $0 < r \leq q$ and take the adjoints knowing that $W(S^*) = \{\overline{\lambda} : \lambda \in W(S)\}$. Letting x to be the unit column vector of $\mathbb{C}^2$ and having it in parametric form as $x = (\theta, \omega, z) = e^{i\omega}[z, e^{i\theta}(1 - z^2)^{\frac{1}{2}}]$, θ and ω being real with $0 \leq z \leq 1$, then as θ traverses the real line, $\langle Sx, x \rangle = e^{i\theta} z(1 - z^2)^{\frac{1}{2}}$ describes a circle with radius $z(1 - z^2)^{\frac{1}{2}}$ and center the origin . Hence,

$$\begin{aligned} W(S) &= \{z(1 - z^2)^{\frac{1}{2}}[qe^{i\theta} + re^{-i\theta}]\} \\ &= \{z(1 - z^2)^{\frac{1}{2}}[(q + r)cos\theta + i(q - r)sin\theta]\} \end{aligned}$$

which describes an origin centered ellipse with major axis of length $q + r$ being horizontal and minor axis of length $q - r$ $(q \neq r)$ being vertical. This implies that the major and minor semi-axes have lengths $\frac{q+r}{2}$ and $\frac{q-r}{2}$ respectively which, from analytic geometry, gives the foci of the ellipse as $\pm((\frac{q+r}{2})^2 - (\frac{q-r}{2})^2)^{\frac{1}{2}} = \pm(qr)^{\frac{1}{2}}$. Therefore by Proposition 3.6, $W(S)$ has its foci being eigenvalues of S. □

The result which follows shows that boundary points of a numerical range of a norm-attaining operator are its eigenvalues if the boundary's curvature is infinite.

Proposition 4.7. *Let $S \in NA(H)$ and λ be a point on $\partial W(S)$ at the part where its curvature is infinite. Then if S is reflexive, λ is its eigenvalue.*

Proof. Let S be reflexive. Then we have $\lambda = \langle S\xi, \xi \rangle$ with $\|\xi\| = 1$. Suppose y is orthogonal to ξ where $\|y\| = 1$. Consider a subspace $\mathcal{V}$ whose span is ξ and y. From Theorem 3.2 the compression $S_{\mathcal{V}}$ of S to $\mathcal{V}$ has its numerical range being a ellipse which is degenerate with $W(S)$ having λ in its boundary. Since $W(S_{\mathcal{V}})$ is contained in $W(S)$ which has no closed disc having λ, it means it does not contain a closed elliptical disc containing λ. This implies that $W(S_{\mathcal{V}})$ is a line segment whose endpoint is λ which is then an eigenvalue of $S_{\mathcal{V}}$ and therefore of S also. □

Proposition 4.8. *Let $S_1 \in NA(H)$. Then the conditions which follow are equivalent*

(i). $0 \in \{\overline{W(S_1 + S_2)} : S_2 \in NA(H)\ is\ of\ finite\ rank\}$.

(ii). $0 \in W_e(S_1)$.

(iii). A unit vector sequence y_n exists in order for y_n to converge to 0 weakly but $\langle S_1 y_n, y_n \rangle$ converges to 0.

(iv). An orthonormal sequence e_k exists such that $\langle S_1 e_k, e_k \rangle \to 0$.

(v). An infinite-dimensional projection P_o exists in order for $P_o S_1 P_o$ to be compact.

Proof. $(i) \Rightarrow (iv)$. Let $\varepsilon_n \to 0$ and $e_1, e_2, ...e_k$ be orthogonal unit vectors such that $|\langle S_1 e_n, e_n\rangle| < \varepsilon_n; n = 1, 2, ..., k$. Being a subspace, suppose V has e_n as its spanning vectors and P_o is a projection on V. To show that there is a unit vector e_{k+1} such that $|\langle S_1 e_{k+1}, e_{k+1}\rangle| < \varepsilon_{k+1}$, we need to show that $0 \in W((I - P_o)S_1)|V^\perp$. For this, take $\lambda \in W((I - P_o)S_1|V^\perp)$ and let $S_2 = \lambda P_o - P_o S_1 P_o - (I - P_o)S_1 P_o - P_o S_1(I - P_o)$. Then S_2 is of finite rank implying that $S_1 + S_2 = \lambda P_o + (I - P_o)S_1(I - P_o) = \lambda I_V \oplus (I - P_o)S_1|V^\perp$. Since $W(S_1 \oplus S_2)$ is generally the convex hull of $W(S_1)$ and $W(S_2)$, then it means that $W(S_1 + S_2) = W((I - P_o)S_1|V^\perp)$ because $W((I - P_o)S_1|V^\perp)$ is convex and contains λ. Thus (i) implies $0 \in W((I - P_o)S_1|V^\perp)$.

We now show that $(iv) \Rightarrow (v)$. Suppose e_k is an orthonormal sequence with $\langle S_1 e_k, e_k\rangle \to 0$. Then assume

$$\sum_{k=1}^{\infty} |\langle S_1 e_k, e_k\rangle|^2\} \quad < \quad \infty \tag{4.2.2}$$

By Bessel's inequality and putting $k_1 = 1$ this means that $\sum\limits_{k=1}^{\infty} |\langle S_1 e_{k_1}, e_k\rangle|^2 \leq \|S_1 e_{k_1}\|^2$ and $\sum\limits_{k=1}^{\infty} |\langle S_1 e_k, e_{k_1}\rangle|^2 \leq \| \ S_1 e_{k_1} \ \|^2$ implying that $k_2 > k_1$ exists so that $\sum\limits_{k=k_2}^{\infty} |\langle S_1 e_{k_1}, e_k\rangle|^2 < \frac{1}{2}$ and $\sum\limits_{k=k_2}^{\infty} |\langle S_1 e_k, e_{k_1}\rangle|^2 < \frac{1}{2}$. Performing this procedure repeatedly generates a positive integers sequence $\{k_n\}$ which is increasing and satisfying the condition

$$\sum_{k=k_{n+1}}^{\infty} |\langle S_1 e_{k_n}, e_k\rangle|^2 < 2^{-n}, \quad \sum_{k=k_{n+1}}^{\infty} |\langle S_1 e_k, e_{k_n}\rangle|^2 < 2^{-n}; \ n \geq 1 \quad . \tag{4.2.3}$$

Inequality 4.2.2 and inequality 4.2.3 imply that $\sum\limits_{k=k+1}^{\infty} |\langle S_1 e_{k_{i}, e_{k_j}}\rangle|^2 < \infty$. Since P_o is a projection on span e_{k_n}, then $P_o S_1 P_o$ is compact as it is then a Hilbert-Schimdt operator. □

For any norm-attaining operator, the numerical range's extreme points can be used to tell whether it is closed or not. This is what the proposition below shows.

Proposition 4.9. *Let $S \in NA(H)$ be normal. Then $W(S)$ is closed if and only if there exist an extreme point which is contained in its closure.*

Proof. Assume the point z is extreme in $W(S)$. Then in $W(S-z) = W(S)-z$, 0 is extreme. Also $\sigma_p(S-z) = \sigma_p(S)-z$. We only need to show that if 0 is extreme in $W(S)$, then in $W(S)$ it is an eigenvalue. Now, since $W(e^{i\theta}S) = e^{i\theta}W(S)$, $\sigma_p(e^{i\theta}S) = e^{i\theta}\sigma_p(S)$ and 0 is extreme in $W(S)$, it can be assumed that the entire $W(S)$, which is convex by Theorem 3.2, is in $Rez \geq 0$. Letting (y, μ) to be some finite measure and $H = L^2(y, \mu)$, S becomes unitarily similar to multiplying $L^2(y, \mu)$ by a function $b(y) \in L^\infty(y, \mu)$ and $b(y) = (Sh)(y)$ $\forall h \in H$. Since $W(S)$ was assumed to lie in $Rez \geq 0$ and by Theorem 3.8, $\sigma(S) \in \overline{W(S)}$, then Re $b(y) \geq 0$. If 0 is assumed not to be an eigenvalue, then $|b(y)| > 0$. Let $D_o = \{y \in X : \text{Im } b(y) \geq 0\}$ while $D_1 = \{y \in X : \text{Im } b(y) < 0\}$ so that since 0 is an extreme point of $W(S)$, we have a h where $\|h\| = 1$ to give

$$\begin{aligned}
\langle Sh, h \rangle &= \int_X b(y)|h(y)|^2 d\mu \\
&= \int_{D_o} b|h|^2 d\mu + \int_{D_o} b|h|^2 d\mu \\
&= \int_X b|\chi D_o h|^2 d\mu + \int_X b|\chi D_1 h|^2 d\mu \\
&= \int_X b|f_1|^2 d\mu + \int_X b|f_2|^2 d\mu \\
&= \langle Sf_1, f_1 \rangle + \langle Sf_2, f_2 \rangle
\end{aligned}$$

where $f_i(c) = \chi D_i(y)(h)$. But Re $\langle Sf_i, f_i \rangle = 0$ as Re $b(y) \geq 0$ and $\langle Sf_1, f_1 \rangle + \langle Sf_2, f_2 \rangle = 0$. Similarly, $\text{Im}\langle Sf_1, f_1 \rangle \geq 0$ and $\text{Im}\langle Sf_2, f_2 \rangle \leq 0$. However,

$\langle Sf_1, f_1\rangle \neq 0$ because if $\langle Sf_1, f_1\rangle = 0$ then $\langle Sf_2, f_2\rangle = 0$ which is impossible as this will imply that h has complementary support and yet $|h| = 1$ with $|b(y)| > 0$, hence $\langle Sf_2, f_2\rangle \neq 0$. Setting $t_i = \frac{f_i(y)}{\|f_i\|}$ and letting $z_i = \langle St_i, t_i\rangle$ we have two points $z_1, z_2 \in W(S)$ on the imaginary axis with 0 being interior point of the line connecting them. This is a contradiction since 0 was assumed to be extreme in $W(S)$ therefore making it an eigenvalue. □

If each set of orthonormal points in a norm-attaining operator's numerical range converge to zero, then the implication is that the operator is compact which is established as follows:

Proposition 4.10. *Let $S \in NA(H)$ and $\langle Sx_n, x_n\rangle \to 0$ for each orthonormal set $\{x_n\}_{n\geq 1} \in H$. Then S is compact.*

Proof. Let $\{x_n\}_{n\geq 1}$ be orthonormal and P_o be a projection onto $\text{Span}(x_1, ..., x_n)$. Then we obtain $\lim_{n\to\infty} \|(I-P_o)S(I-P_o)\| = 0$. But

$$\begin{aligned} |\langle Sx_n, x_n\rangle| &= |\langle (I-P_o)S(I-P_o)x_n, x_n\rangle| \\ &\leq \|(I-P_o)S(I-P_o)\| \end{aligned}$$

which implies that $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = 0$ for each orthonormal set. Since with $x, y \in H$ being unit vectors, we have $|\langle Sx, y\rangle| = \frac{\|S\|}{2}$ and by polarization identity, there exists a k so that $\frac{\|S\|}{2} \leq |\langle Sx, y\rangle| \leq \frac{1}{4}\{\sum_{k=1}^{4} |i^k \langle S(x+i^k y), x+i^k y\rangle|$ which implies that $|\langle S(x+i^k y), x+i^k y\rangle| \geq \frac{\|S\|}{2}$. But $\|x+i^k y\| \leq 2$ means a unit vector η_1 exists to give $|\langle S\eta_1, \eta_1\rangle| \geq \frac{\|S\|}{8}$. Letting P_1 be an orthogonal projection onto the $\text{Span}(\eta_1)$ and repeating the process, we obtain that a unit vector η_2 which is orthogonal to η_1 exists such that $|\langle S\eta_2, \eta_2\rangle| \geq \frac{\|(I-P_1)S(I-P_1)\|}{8}$. Therefore, proceeding this way recursively, an orthonormal set $\{\eta_n\}_{n=1}^{\infty}$ exists such that P_n

being the projection onto Span$\{\eta_1, ..., \eta_n\}$ then $|\langle S\eta_n, \eta_n\rangle| \geq \frac{\|(I-P_{n-1})S(I-P_{n-1})\|}{8}$ and since $\lim_{n\to\infty}\langle S\eta_n, \eta_n\rangle = 0$, then $\lim_{n\to\infty} \|(I-P_n)S(I-P_n)\| = 0$ hence S is compact being a limit of $-P_nS - SP_n + P_nSP_n$ which is of finite rank. $\square$

Proposition 4.11. *Let $S \in NA(H)$. Then the statements which follow are equivalent:*

(i). $\lambda \in W(S)$;

(ii). $\exists$ a sequence $(\xi_n)_{n=1}^{\infty} \in H$ which is orthonormal such that $\lim_{n\to\infty}\langle S\xi_n, \xi_n\rangle = 0$.

Proof. Let the orthonormal sequence (ξ_n) exist such that $\lim_{n\to\infty}\langle S\xi_n, \xi_n\rangle = 0$. Then by Proposition 4.10 and for each compact operator K, $\lambda = \lim_{n\to\infty}\langle S\xi_n, \xi_n\rangle = \lim_{n\to\infty}\langle (S+K)\xi_n, \xi_n\rangle \in \overline{W(S+K)}$. Hence, $\lambda \in W_e(S)$ since K was arbitrary and $\lambda \in \overline{W(S)}$ implies a unit vector ξ_1 exists such that $|\langle S\xi_1, \xi_1\rangle - \lambda| \leq \frac{1}{2}$. Let span $\{\xi_1\} = L_1$ and the orthogonal projection onto L_1 be P_1. Also let $\eta_1 \in W(I-P_1)S|_{L_1^\perp}$ and $G_1 = \eta_1 P_1 - P_1SP_1 - P_1S(I-P_1) - (I-P_1)SP_1$. This clearly shows that G_1 is of finite rank on H which therefore means it is compact. Hence $\lambda \in \overline{W(S+G_1)} = \overline{W(\eta_1 P_1 + (I-P_1)S(I-P_1))}$. But clearly, $W(\eta_1 P_1 + (I-P_1)S(I-P_1)) = \{\langle(\eta_1 P_1 + (I-P_1)S(I-P_1))\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}$ $= \{\eta_1\|\xi_1\|^2 + \langle((I-P_1)S(I-P_1))\xi_2, \xi_2\rangle : \xi_1 \in L_1, \xi_2 \in L_1^\perp, \|\xi_1\|^2 + \|\xi_2\|^2 = 1\}$ $= \{\eta_1\|\xi_1\|^2 + \|\xi_2\|^2\langle((I-P_1)S(I-P_1))\frac{1}{\|\xi_2\|}\xi_2, \frac{1}{\|\xi_2\|}\xi_2\rangle : \xi_1 \in L_1, \xi_2 \in L_1^\perp, \|\xi_1\|^2 + \|\xi_2\|^2 = 1\}$. But since $\eta_1 \in W((I-P_1)S|_{L_1^\perp})$ and $W((I-P_1)S|_{L_1^\perp})$ is convex by Toeplitz-Hausdorff Theorem 3.2, we have $W(\eta_1 P_1 + (I-P_1)S(I-P_1)) = W((I-P_1)S|_{L_1^\perp})$. Therefore, $\lambda \in W((I-P_1)S|_{L_1^\perp})$ and a unit vector $x_2 \in L_1^\perp$ exists which is orthogonal to x_1 giving $|\langle Sx_2, x_2 - \lambda\rangle| \leq \frac{1}{2^2}$. Taking $x_1, ..., x_n$ to be orthonormal vectors such that $|\langle Sx_n, x_n - \lambda\rangle| \leq \frac{1}{2^n}$ and repeating this procedure with $L_n = \text{Span}\{x_1, ..., x_n\}$, P_n the orthogonal projection onto L_n, $\eta_n \in W((I - P_n)S|_{L_n^\perp})$ and $G_n = \eta_n P_n - P_n S P_n - P_n S(I - P_n) - (I - P_n)S P_n$ we obtain a unit vector x_{n+1} which is orthogonal to every x_j, $1 \leq j \leq n$ such that $|\langle S x_{n+1}, x_{n+1}\rangle - \lambda| \leq \frac{1}{2^{n+1}}$. Recursively, this means there is a sequence $(x_n)_{n\geq 1}^{\infty}$ which is orthonormal giving $\lim_{n\to\infty}\langle S x_n, x_n\rangle = \lambda$. $\square$

4.3 Spectra in $NA(H)$

Results on spectra for norm-attaining operators are given here starting with the following.

Proposition 4.12. *Let $S \in NA(H)$. Then the statements which follow are equivalent:*

(i). $\sigma(S) \subset H^+$.

(ii). An invertible norm-attaining operator B exists such that $\overline{W(B^{-1}SB)} \subset H^+$.

(iii). An invertible norm-attaining operator C which is positive exists in order for $\overline{W(C^{-1}SC)} \subset H^+$.

(iv). An invertible norm-attaining operator C which is positive exists in order for $\overline{W(SC)} \subset H^+$.

Proof. $(i) \Rightarrow (ii)$. Follows since $\sigma(S)$ is in the interior of a subset of an open set which is convex and $\overline{W(B^{-1}SB)}$ is in the set. $(iii) \Rightarrow (ii)$ is trivial. By polar decomposition, let $B = P_o U_o$ with P_o being invertible and positive and U_o being unitary. This gives $W(P_o^{-1} S P_o) = W(U_o^T P_o^T S P_o U_o) = W(B^{-1}SB)$

which is, under unitary transformations, invariant. $(iii) \Rightarrow (iv)$ follows due to the identity $\langle SP_o^2 y, y\rangle = \langle (P_o^{-1}SP_o), (P_o y)\rangle$ showing that, for some $\delta > 0$, $W(SP_o^2) \subset \{z : Re\ z \geq \frac{\delta}{\|P_o^{-1}\|^2}\}$ whenever $\overline{W(P_o^{-1}SP_o)} \subset \{z : Re\ z \geq \delta\}$. $(iv) \Rightarrow (i)$. To show this, we apply theorem 3.7 to have $\sigma(S) = \sigma(SP_oP_o^1) \subset \frac{\overline{W(SP_o)}}{\overline{W(P_o)}}$ and $\frac{\overline{W(SP_o)}}{\overline{W(P_o)}} \subset H^+$ which together with the fact that the positive real axis contains $\overline{W(P_o)}$ implies $\sigma(S) \subset H^+$. □

If an operator is equivalent to a norm-attaining operator then the convex hull of the operator's spectrum is the same as the closure of its numerical range as established in the result which follows:

Proposition 4.13. *Let $S \in NA(H)$. Then $co(\sigma(S)) = \bigcap\{\overline{W(TST^{-1})} :$ T is invertible on $H\}$.*

Proof. Suppose $\lambda \notin co(\sigma(S))$. We need to show that $T \in NA(H)$ which is invertible exists such that $\lambda \notin \overline{W(T^{-1}ST)}$. Since $co(\sigma(S))$ is compact, then an open disc Θ containing this exists with λ not in its closure. With the existence of Θ, particularly that $r(S) < 1$, Corollary 3.3 implies the existence of $T \in NA(H)$ which is invertible giving $\|T^{-1}ST\| \leq \frac{(1+r(S))}{2} < 1$ which means that $\overline{W(T^{-1}ST)} \subset \Theta$ and hence $\lambda \notin \overline{W(T^{-1}ST)}$. □

For a closed norm-attaining operator, the residual spectrum and the point spectrum being equal implies that the operators's spectrum is equal to its adjoint's spectrum under certain conditions established in the result which follows.

Proposition 4.14. *Let $S \in NA(H)$ be closed on H. Then $\sigma_r(S) = \sigma_p(S^*)$ iff $\sigma(S) = \sigma(S^*)$ and $R(\lambda_o, S)^* = R(\lambda_o, S^*)$ for all λ_o in $\rho(S)$.*

Proof. By Corollary 3.5 the set $(\lambda_o I - S)D(S)$ cannot be dense in H if some vector $y^* \in H^* \setminus \{0\}$ exists so that we have, for each $x \in D(S)$, $\langle (\lambda_o I - S)x, y^* \rangle = 0$, which is similar to $\langle Sx, y^* \rangle = \langle x, \lambda_o y^* \rangle$. This means that y^* is in $D(S^*) \setminus \{0\}$ and $S^* y^* = \lambda_o y^*$, hence $\lambda_o \in \sigma_p(S^*)$.

Conversely, let $\lambda_o \in \rho(S)$ and take $y \in D(S)$, $x^* \in H^*$ and set $y^* = R(\lambda_o, S)^* x^*$. Then $\langle (\lambda_o I - S)x, y^* \rangle = \langle R(\lambda_o, S)(\lambda_o I - S)x, x^* \rangle = \langle x, x^* \rangle$. Thus, $y^* \in D(S^*)$ with $x^* = (\lambda_o I - S)^* y^* = (\lambda_o I - S^*) y^*$ which implies that $\lambda_o I - S^*$ is surjective. Now taking $x^* \in D(S^*)$ for $x \in H$ and using the fact that $R(\lambda_o, S)y \in D(S)$ we have

$$\begin{aligned} \langle y, R(\lambda_o, S)^* (\lambda_o I - S^*) x^* \rangle &= \langle R(\lambda_o, S)x, (\lambda_o I - S^*) x^* \rangle \\ &= \langle (\lambda_o I - S) R(\lambda_o, S) x, x^* \rangle \\ &= \langle y^*, y \rangle. \end{aligned}$$

This means that $R(\lambda_o, S)^* (\lambda_o I - S^*) x^* = x^*$ and hence $\lambda_o I - S^*$ is injective. Therefore $R(\lambda_o, S^*) = R(\lambda_o, S)^*$.

Similarly, let $\lambda_o \in \rho^*(S)$ and take $x \in D(S)$ so that given each $x^* \in H^*$, it follows that

$$\begin{aligned} \langle (\lambda_o I - S)x, R(\lambda_o, S^*) x^* \rangle &= \langle x, (\lambda_o I - S^*) R(\lambda_o, S^*) x^* \rangle \\ &= \langle x, x^* \rangle. \end{aligned}$$

By Corollary 3.5, $\langle x, y^* \rangle = \|y\|$ whenever y^* is a unit vector giving $\|y\| = \langle (\lambda_o I - S)x, R(\lambda_o, S^*) y^* \rangle \leq \|R(\lambda_o, S^*)\| \|\lambda_o x - Sx\|$ which implies that $\lambda_o \notin \sigma_{ap}(S)$ and $\lambda_o \notin \sigma_p(S^*) = \sigma_r(S)$, hence $\lambda_o \in \rho(S)$. □

The spectrum of a norm-attaining operator is bounded under certain conditions as established in the following.

Theorem 4.15. *Let $S \in NA(H)$ and $\|S\| < |\lambda|$. Then $\sigma(S)$ is bounded.*

Proof. Define $R_{\lambda,j} \in NA(H)$ by $R_{\lambda,j} = -\frac{1}{\lambda}\sum_{n=0}^{j}\frac{S^n}{\lambda^n}$. Since $\frac{\|S\|}{|\lambda|} < 1$, then $\sum_{n=0}^{\infty}\frac{\|S\|^n}{|\lambda|^n}$ is a convergent geometric series. Therefore $R_{\lambda,j}$ is Cauchy and converges to some $A_\lambda \in NA(H)$. So

$$\begin{aligned}
\|A_\lambda(S-\lambda I) - I\| &\leq \|A_\lambda(S-\lambda I) - R_{\lambda,j}(S-\lambda I)\| + \|R_{\lambda,j}(S-\lambda I) - I\| \\
&\leq \|A_\lambda - R_{\lambda,j}\|\|S-\lambda I\| + \| - \frac{S}{\lambda}\sum_{n=0}^{j}\frac{S^n}{\lambda^n} + \sum_{n=0}^{j}\frac{S^n}{\lambda^n} - I\| \\
&= \|A_\lambda - R_{\lambda,j}\|\|S-\lambda I\| + \|\frac{S^{j+1}}{\lambda^{j+1}}\| \\
&\leq \|A_\lambda - R_{\lambda,j}\|\|S-\lambda I\| + (\frac{\|S\|}{|\lambda|})^{j+1},
\end{aligned}$$

which tends to 0 as $j \to \infty$. Hence, $A_\lambda(S-\lambda I) = I$ and

$$\begin{aligned}
\|(S-\lambda I)A_\lambda - I\| &\leq \|(S-\lambda I)A_\lambda - (S-\lambda I)R_{\lambda,j}\| + \|(S-\lambda I) - I\| \\
&\leq \|S-\lambda I\|\|A_\lambda - R_{\lambda,j}\| + \|\frac{\|S\|}{|\lambda|}\|^{j+1}
\end{aligned}$$

where $(S-\lambda I)A_\lambda = I$ implying that $A_\lambda = (S-\lambda I)^{-1}$. Thus if $\|S\| < |\lambda|$, then $\lambda \in \rho(S)$ and therefore $\sigma(S)$ is in the disc $|\lambda| \leq \|S\|$ which means it is bounded as required. □

Given a norm-attaining operator, its spectrum is a set which is closed in the complex plane as proved in this result:

Proposition 4.16. *Let $S \in NA(H)$. Then $\sigma(S) \subseteq \mathbb{C}$ is closed.*

Proof. Suppose $\lambda \in \rho(S)$ and $(\kappa - \lambda) < \|R_\lambda\|^{-1}$. Let $R_{\kappa,j} \in NA(H)$ be defined by $R_{\kappa,j} = R_\lambda \sum_{n=0}^{j}(\kappa-\lambda)^n R_\lambda^n$. Since $|(\kappa-\lambda)| < \|R_\lambda\|^{-1}$, then $R_{\lambda,\kappa}$ is Cauchy converging to some $A_\kappa \in NA(H)$. Therefore, since $R_\lambda = (S - \lambda I)^{-1}$ we get

$$\begin{aligned}
\|A_\kappa(S - \kappa I) - I\| &\leq \|A_\kappa(S - \kappa I) - R_{\kappa,j}(S - \kappa I)\| + \|R_{\kappa,j}(S - \kappa I + \lambda I - \lambda I) - I\| \\
&\leq \|A_\kappa - R_{\kappa,j}\|\|S - \kappa I\| + \|\sum_{n=0}^{j}(\kappa - \lambda)^n R_\lambda^n \\
&\quad -(\kappa - \lambda)R_\lambda \sum_{n=0}^{j}(\kappa - \lambda)^n R_\lambda^n - I\| \\
&= \|A_\kappa - R_{\kappa,j}\|\|S - \kappa I\| + \| - (\kappa - \lambda)^{j+1} R_\lambda^{j+1}\| \\
&= \|A_\kappa - R_{\kappa,j}\|\|S - \kappa I\| + |\kappa - \lambda|^{j+1} R_\lambda^{j+1}
\end{aligned}$$

and this tends to zero as $j \to \infty$. Thus $A_\kappa(S - \kappa I) = I$.
Similarly $(S - \kappa I)A_\kappa = I$ which implies that $(S - \kappa I)^{-1} = A_\kappa$. Therefore, $\kappa \in \rho(S)$ meaning $\rho(S)$ is open. Hence, $\sigma(S)$ is closed. □

If a norm-attaining operator S is self-adjoint, then its spectrum lies in the interval $[-\|S\|, \|S\|]$ as shown in the proposition below.

Proposition 4.17. *Let $S \in NA(H)$ and $S = S^*$. Then $\sigma(S) \in \mathbb{R}$ and $\sigma(S) \subseteq [-\|S\|, \|S\|]$.*

Proof. Since $r(S) \leq \|S\|$, then we need only to show that $\sigma(S) \in \mathbb{R}$. Suppose $\lambda_o = \beta + i\theta \in \mathbb{C}$ $\beta, \theta \in \mathbb{R}$ with $\theta \neq 0$. Then given $\pi \in H$ we get

$$\begin{aligned}
\|(S - \lambda_o I)\pi\|^2 &= \langle (S - \lambda_o I)\pi, (S - \lambda_o I)\pi \rangle \\
&= \langle (S - \beta I)\pi, (S - \beta I)\pi \rangle + \langle (-i\theta)\pi, (-i\theta)y \rangle \\
&\quad + \langle S\pi, (-i\theta)\pi \rangle + \langle (-i\theta)y, S\pi \rangle \\
&= \|(S - \beta I)\pi\|^2 + \theta^2\|\pi\|^2 \\
&\geq \theta^2\|\pi\|^2.
\end{aligned}$$

This estimate implies that $S - \lambda_o I$ is a one-to-one operator whose range is closed. Now suppose range$(S_o - \lambda I) \neq H$, then λ_o is in the residual spectrum of S such that $\overline{\lambda_o} = \beta - i\theta$ is an eigenvalue of S implying that S has an eigenvalue which is not a real number hence contradicting the property that eigenvalues of bounded self-adjoint operators are real. Therefore λ_o is in $\rho(S)$ since it is not real. $\square$

Proposition 4.18. *Let $S \in NA(H)$ and $p_1(\lambda) = \beta_0 + \beta_1\lambda + ... + \beta_n\lambda^n$ be a polynomial. If $p_2(S) = \beta_0 + \beta_1\varsigma + ... + \beta_n\varsigma^n$, then $\sigma(p_2(S)) = p_1(\sigma(S)) = \{p_1(\lambda) : \lambda \in \sigma(S)\}$.*

Proof. Since $n = 0$ is obvious, we let $n \geq 1$ and take $\lambda_0 \in \sigma(S)$ such that $\lambda_0 - S$ is not invertible. Then $p_1(\lambda_0) - p_2(S)$ is not invertible since

$$\begin{aligned} p_1(\lambda_0) - p_2(S) &= \sum_{k=0}^{n} \beta_k(\lambda_0^k - S^k) \\ &= \sum_{k=1}^{n} \beta_k(\lambda_0^k - S^k) \\ &= (\lambda_0 - S)\sum_{k=1}^{n} \beta_k \sum_{i=1}^{k-1} \lambda_0^{k-i} S^{i-1} \end{aligned}$$

with $\lambda_0 - S$ and $\sum_{k=1}^{n} \beta_k \sum_{i=1}^{k-1} \lambda_0^{k-i} S^{i-1}$ commuting. This implies that $p_1(\sigma(S)) \subset \sigma(p_2(S))$. Also, if $\mu \notin \{p_1(\lambda) : \lambda \in \sigma(S)\}$ and $\lambda_1, \lambda_2, ..., \lambda_n$ are solutions to the polynomial $\mu - p_1(\lambda)$ then $\lambda_1, \lambda_2, ..., \lambda_n \notin \sigma(S)$. Additionally, $\mu - p_1(\lambda) = \xi(\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2}...(\lambda_n - \lambda)^{m_n}$ where $m_1, m_2, ...m_n \in \mathbb{N}$ and $\xi \neq 0$ which means that
$\mu - p_2(S) = \xi(\lambda_1 - S)^{m_1}(\lambda_2 - S)^{m_2}...(\lambda_n - S)^{m_n}$.
Hence $\mu - P_2(S)$ is invertible(being a product of invertible operators) and therefore $\mu \notin \sigma(P(S_2))$. This shows that $p_1(\sigma(S)) \supset \sigma(p_2(S))$ as required. $\square$

A norm-attaining operator which is dense and closed on H has its spectrum closed in the complex plane as established in the result below.

Proposition 4.19. *Let $S \in NA(H)$. If S is dense and closed on H then $\sigma(S)$ is closed in $\mathbb{C}$.*

Proof. Let $\lambda_0 \in \rho(S)$. Then $|\lambda_0 - \lambda| < \|S - \lambda_0\|^{-1}$ for $\lambda \in \mathbb{C}$ and the series $\sum_{n=0}^{\infty}(\lambda_0 - S)^{-n-1}(\lambda_0 - \lambda)^n$ converges in H to δ. We now show that for $\varphi \in D(\lambda - S) = D(S)$ we have $\delta(\lambda - S)\varphi = \varphi$ and that for $\varsigma \in H$ we have $\delta\varsigma \in D(S)$ and $(\lambda - S)\delta\varsigma = \varsigma$ which will imply that $\delta = (\lambda - S)^{-1}$. Now let $\varphi \in D(S)$. Then $(\lambda - S)\varphi = (\lambda - \lambda_0)\varphi + (\lambda_0 - S)\varphi$ and hence

$$\begin{aligned}\delta(\lambda - S)\varphi &= -\sum_{n=0}^{\infty}(\lambda - \lambda_0)^{n+1}(\lambda_0 - S)^{-n-1}\varphi \\ &\quad + \sum_{n=0}^{\infty}(\lambda - \lambda_0)^{n}(\lambda - S)^{-n}\varphi = \varphi.\end{aligned}$$

Furthermore for any integer $n > 0$ we have $(\lambda_0 - S)^{-n-1}\varsigma \in D(S)$ and so taking $\varsigma_N = \sum_{n=0}^{N}(\lambda_0 - \lambda)^n(\lambda_0 - S)^{-n-1}\varsigma$ gives the sequence $(\varsigma_k)_{k\in\mathbb{N}}$ of elements in $D(\lambda - S)$ which converges to $\delta\varsigma$. Moreover,

$$\begin{aligned}(\lambda - S) &= ((\lambda - \lambda_0) + (\lambda_0 - S))\varsigma_N \\ &= (\lambda - \lambda_0)\varsigma_N + \sum_{n=0}^{N}(\lambda_0 - \lambda)^n(\lambda_0 - S)^{-n}\varsigma \\ &= (\lambda - \lambda_0)\varsigma_N + (\lambda_0 - \lambda)\sum_{n=0}^{N}(\lambda_0 - \lambda)^{n-1}(\lambda_0 - S)^{-n}\varsigma \\ &= (\lambda - \lambda_0)\varsigma_N + (\lambda_0 - \lambda)\left(\frac{1}{\lambda_0 - \lambda}\varsigma + \sum_{i=0}^{N-1}(\lambda_0 - \lambda)^i(\lambda_0 - S)^{-i-1}\varsigma\right) \\ &= (\lambda - \lambda_0)\varsigma_N + (\lambda_0 - \lambda)\left(\frac{1}{\lambda_0 - \lambda}\varsigma + \varsigma_{N-1}\right) \\ &= \varsigma + (\lambda - \lambda_0)(\varsigma_N - \varsigma_{N-1}).\end{aligned}$$

But $\lim_{N\to\infty}\{\varsigma + (\lambda - \lambda_0)(\varsigma_N - \varsigma_{N-1})\} = \varsigma$, hence by the closedness of $(\lambda - S)$

we get that $\delta\varsigma \in D(\lambda - S)$ and $(\lambda - S)\delta\varsigma = \varsigma$ which means that $(\lambda - S)$ is invertible and therefore λ is in $\rho(S)$. This then gives $\rho(S)$ as an open $\mathbb{C}$ implying $\sigma(S) = \mathbb{C} \setminus \rho(S)$ as required. □

4.4 Norms in $NA(H)$

Theorem 4.20. *Let $S_o \in NA(H)$ and $S_o = S_o^*$. If $\varphi_{o_1}(g) \in (H)$ is the unique element that norms some $g \in H$ and that $\varphi_{o_2}(S_o g) \in H$, then $\mu \in \mathbb{R}$ exists in order for $S_o^*(\varphi_{o_2}(S_o g)) = \mu\varphi_{o_1}(g)$ and $\|S_o\| = \mu$.*

Proof. Let $\Gamma_{o_1}, \Gamma_{o_2} \in (H)$ be defined as $\Gamma_{o_1}(f_o) = \langle f_o, \varphi_{o_1}(g)\rangle$, $\Gamma_{o_2}(f_o) = \frac{1}{\|S_o\|}\langle S_o f_o, \varphi_{o_2}(g)\rangle = \frac{1}{\|S_o\|}\langle f_o, S^*(\varphi_{o_2}(S_o g))\rangle$ for some $f_o \in H$. Then $\|\Gamma_{o_1}\| = 1$ (since $\|\varphi_{o_1}(g)\| = 1$) and $\Gamma_{o_1}(g) = \|g\|$, so Γ_{o_1} norms g. Similarly $\|\varphi_{o_2}(S_o g)\| = 1$ implies that $\|\Gamma_{o_2}\| \leq 1$, but using $\|S_o g\| = \|S_o\|\|g\|$ we have $\Gamma_{o_2}(g) = \|g\|$ which means that g is normed by Γ_{o_2}. Now, since (H) is smooth, then $\Gamma_{o_1} = \Gamma_{o_2}$ and therefore $S_o^*(\varphi_{o_2}(S_o g)) = \mu\varphi_{o_1}(g)$ with $\mu = \|S_o\|$ as required. □

Corollary 4.21. *Let $NA(H)_1 = L^q(X,\mu)$, $NA(H)_2 = L^p(Y,W)$ where $q, p \in (1,\infty)$ with h being a solution in Theorem 4.20 . Let $S \in NA(H)$ in order for $S : NA(H)_1 \to NA(H)_2$. Then S has h as its critical point and $\beta = \lambda^q\|h\|_p^{q-p}$. Moreover $\|S\| = \lambda^q$.*

Proof. We first note that if h satisfies Theorem 4.20, then we have $\|Sh\|_p = \langle Sh, \varphi_{o_2}(Sh)\rangle = \langle h, S^*\varphi_{o_2}(Sh)\rangle = \langle h, \lambda\varphi_{o_1}(h)\rangle = \lambda\|h\|_q$. Substituting $\varphi_{L^q}(h) = \|h\|_p^{-(q-1)}\text{sgn}(h)|h|^{q-1}$ into $S^*(\varphi_{o_2}(Sh)) = \lambda\varphi_{o_1}(h)$ and multiplication by $\|Sh\|_p^{p-1}$ gives

$S^*((Sh)|Sh|^{q-1}) = \lambda\|Sh\|_p^{p-1}\|h\|_p^{-(q-1}(h)|h|^{q-1} = \lambda^p\|h\|_q^{p-q}(h)|h|^{q-1}$ as required. □

Proposition 4.22. *Let $S \in NA(H)$ be bounded and $\|S\| < 1$. Then $(I - S)$ is bounded with an inverse equal to $\sum_{k=0}^{\infty} S^k$ i.e $\lim_{n\to\infty} \sum_{k=0}^{n} S^k$ is norm convergent to $(I - S)^{-1}$.*

Proof. Given $\|S\| < 1$ then $\sum_{k=0}^{n} \|S\|^k$ is a geometric series which is convergent and $\sum_{k=0}^{n} \|S^k\| < \sum_{k=0}^{n} \|S\|^k$ then implies the sequence $\sum_{k=0}^{n} \|S^k\|$ is norm cauchy and therefore converges to a bounded operator as the range of S is a Banach space. Now as a power series, $(I - S)\sum_{k=0}^{\infty} S^k = \sum_{k=0}^{\infty}(S^k - S^{k+1}) = I$ since it is norm convergent. Likewise, $(\sum_{k=0}^{\infty} S^k)(1 - S) = \sum_{k=0}^{\infty}(S^k - S^{k+1}) = I$ which shows that the series is equal to $(I - S)^{-1}$ hence completing the proof. □

Proposition 4.23. *Let $S \in M_n$ be norm attaining and $\psi : M_n \to \mathbb{C}^n$ be a contractive linear mapping so that $\psi(S) = \|S\|$. Then $\exists$ a vector $\xi \in \mathbb{C}^n$ where $\|\xi\| = 1$ and $\|S\xi\| = \|S\|$ with $\langle S\xi, \xi\rangle = \overline{\psi(I)}\|S\|$.*

Proof. Let $S \neq 0$ and $\|S\| = 1$ (or else S can be replaced by $\frac{S}{\|S\|}$). Riesz representation Theorem 3.1 requires that a matrix $C \in M_n$ exists so that we have $\psi(Q) = trace\ (CQ)\ \forall Q \in M_n$. Letting $C = P_oV$ where $P_o \in M_n^+$ and V is unitary, $trace\ Q = \psi(V^*) \leq \|V^*\| = 1 = \psi(S) = trace\ (P_oVS)$. Let $Q = V^*P_o^{\frac{1}{2}}$ and $A = SP_o^{\frac{1}{2}}$. Then $trace\ (Q^*A) = trace\ (P_oVS) = 1$, $trace\ (Q^*Q) = trace\ (P_o) \leq 1$, $trace\ (A^*A) = trace\ (P_o^{\frac{1}{2}}S^*SP_o^{\frac{1}{2}}) \leq trace\ (P_o) \leq 1$ and therefore $trace\ (Q - A)^*(Q - A) \leq 0$. Hence $Q = A$, $trace\ (P_o) = trace\ (Q^*Q) = trace\ (Q^*A) = 1$ and $trace\ (SP_o) = trace\ (V^*P_o) = \overline{\psi(I)}$. Since $P_o = P_o^{\frac{1}{2}}Q^*QP_o^{\frac{1}{2}} = P_o^{\frac{1}{2}}S^*SP_o^{\frac{1}{2}}$, then it means range P_o is an m-dimensional linear subspace $u \in \mathbb{C}^n : \|Su\| = \|u\|$ with $1 \leq m \leq n$. If B is a matrix whose order is $n \times m$ so that $B^*B = I_k$ where BB^* is a projection onto range P_o, we have $P_oBB^* = P_o = BB^*P_o$ where B^*P_oB is a matrix belonging to a compact convex set $\mathcal{D} = \{R \in M_k^+ : trace\ R = 1, trace\ (B^*SBR) = \overline{\psi(I)}\}$ found through the intersection between three real hyperplanes and the $k \times k$ hermitian matrices, M_k^+. $\mathcal{D}$ contains a rank-1 matrix xx^* where $x \in \mathbb{C}^k$ with the properties $\langle B^*SBx, x\rangle = \overline{\psi(I)}$ and $\|x\|^2 = 1 = tr\ (xx^*)$. If we take $y = Bx$, all the required conditions for y are obtained. □

Proposition 4.24. *Let $S_1, S_2 \in NA(H)$ be nonzero $k \times k$ matrices. Then the statements which follow are equivalent.*

(i). $\exists$ *unit vectors* $x, \xi \in \mathbb{C}^k$ *such that* $\|S_1x\| = \|S_1\|$, $\|S_2\xi\| = \|S_2\|$ *and* $\frac{\langle S_1x,x\rangle}{\|S_1\|} = \frac{\langle S_2\xi,\xi\rangle}{\|S_2\|}$.

(ii). $\exists$ *unit vectors* $x, \xi \in \mathbb{C}^k$ *such that* $\|S_1x\| = \|S_1\|$, $\|S_2\xi\| = \|S_2\|$ *and* $\|S_1x\| + \|S_2\xi\| \leq \|(S_1 + \mu I)x\| + \|(S_2 - \mu I)\xi\|\ \forall \mu \in \mathbb{C}$.

(iii). $\|S_1\| + \|S_2\| \leq \|S_1 + \mu I\| + \|S_2 - \mu I\|\ \forall \mu \in \mathbb{C}$

Proof. $(i) \Rightarrow (ii)$:Suppose x and ξ satisfy condition (i). Then

$$\begin{aligned}
\|S_1x\| + \|S_2\xi\| &= \left\| \begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} \langle S_2\xi, \xi\rangle \\ \{\|S_2\xi\|^2 - |\langle S_2\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \right\| \\
&= \| \begin{pmatrix} \langle (S_1 + \mu I)x, x\rangle \\ \{\|(S_1 + \mu I)x\|^2 - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \\
&\quad + \begin{pmatrix} \langle (S_2 - \mu I)\xi, \xi\rangle \\ \{\|(S_2 - \mu I)\xi\|^2 - |\langle (S_2 - \mu I)\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \| \\
&\leq \left\| \begin{pmatrix} \langle (S_1 + \mu I)x, x\rangle \\ \{\|(S_1 + \mu I)x\|^2 - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \right\| \\
&\quad + \left\| \begin{pmatrix} \langle (S_2 - \mu I)\xi, \xi\rangle \\ \{\|(S_2 - \mu I)\xi\|^2 - |\langle (S_2 - \mu I)\xi, \xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \right\| \\
&= \|(S_1 + \mu I)x\| + \|(S_2 - \mu I)\xi\|
\end{aligned}$$

$(ii) \Rightarrow (iii)$ as the vectors x and ξ are unit. $(iii) \Rightarrow (i)$:Considering $(M_k \times M_k, u)$ which is a linear normed space with $u(X_1, X_2) = \|X_1\| + \|X_2\|$, we have a contractive linear functional ψ with respect to u on span $\{(S_1, S_2), (I, -I)\}$ defined by $\psi(S_1, S_2) = \|S_1\| + \|S_2\|$ and $g(I, -I) = 0$ if and only if (iii) holds. Now ψ is extended to a contractive linear functional Ψ on $M_k \times M_k$ by Hahn Banach Theorem to get $\|S_1\| + \|S_2\| = \Psi(S_1, S_2) \leq |\Psi(S_1, 0)| + |\Psi(0, S_2)| \leq \|S_1\| + \|S_2\|$ which gives $\Psi(S_1, 0) = \|S_1\|$ and $\Psi(0, S_2) = \|S_2\|$. By Proposition 4.23 and given that $X_1 \mapsto \Psi(X_1, 0)$ is contractive we have a unit vector $x \in \mathbb{C}^n$ giving $\|S_1x\| = \|S_1\|$ and $\frac{\langle S_1x, x\rangle}{\|S_1\|} = \overline{\Psi(I, 0)}$. Likewise, we have a complex unit vector ς such that $\|S_2\varsigma\| = \|S_2\|$ and $\frac{\langle S_2\varsigma, \varsigma\rangle}{\|S_2\|} = \overline{F(0, I)}$. Finally, since $\Psi(I, -I) = \Psi(I, 0) - \Psi(0, I)$ then $F(I, 0) = F(0, I) = 0$ and therefore (i) is true. □

Lemma 4.25. *Let* $S_1, S_2 \in NA(H)$*. Then* $\sup\{\|U^*S_1U+V^*S_2V\|\} = min\{\|S_1+\mu I\| + \|S_2 - \mu I\|\}$ *where* U,V *are unitaries and* $\mu \in \mathbb{C}$*. Additionally the above equality is equivalent to* $\sup\{\|S_1A + S_2A\| : A \in NA(H), \|A\| \leq 1\}$ *and further equivalent to* $\sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}$.

Proof. Let $\beta \in \mathbb{C}$ and $a \in [0, \infty)$. Suppose $x, x^{'} \in H$ are unit vectors where $\langle x, x^{'}\rangle = 0$ giving $S_1x = \beta + ax^{'}$ with x uniquely determining $x^{'}$ for all $a \neq 0$. Then
$\begin{pmatrix} \beta \\ a \end{pmatrix} = \begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix}$ is a vector in $\mathbb{C} \times \mathbb{R}$ whose length is $\|S_1x\|$. Let

$$\Phi(S_1) = \begin{pmatrix} \langle S_1x, x\rangle \\ \{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \subseteq \mathbb{C} \times [0, \infty) \qquad (4.4.1)$$

It should be noted that $\Gamma(S_1 + \mu I) = \{x + \begin{pmatrix} \mu \\ 0 \end{pmatrix} : x \in \Gamma(S_1)\}$ since

$$\{\|S_1x\|^2 - |\langle S_1x, x\rangle|^2\}^{\frac{1}{2}} = \{\|(S_1 + \mu I)x\| - |\langle (S_1 + \mu I)x, x\rangle|^2\}^{\frac{1}{2}}.$$

Hence $\Gamma(S_1) + \Gamma(S_2) = \Gamma(S_1 + \mu I) + \Gamma(S_2 - \mu I)$ which therefore means
$\sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}$
$= \sup\{\|x+x\prime\| : x \in \Gamma(S_1+\mu I), x\prime \in \Gamma(S_2-\mu I)\} \forall \mu \in \mathbb{C}$. Now taking $e_1, e_2 \in H$ be unit vectors such that $\langle e_1, e_2\rangle = 0$ and letting $x \in \Gamma(S_1)$, $x^{'} \in \Gamma(S_2)$, then we have $U, V \in NA(H)$ as unitary operators to give $x = \begin{pmatrix} \langle U^*S_1Ue_1, e_1\rangle \\ \langle U^*S_1Ue_1, e_2\rangle \end{pmatrix}$

and $x^{'} = \begin{pmatrix} \langle V^*S_2Ve_1, e_1\rangle \\ \langle V^*S_2Ve_1, e_2\rangle \end{pmatrix}$. This gives

$$\|x+x^{'}\| = \|(U^*S_1U + V^*S_2V)e_1\| \leq \|U^*S_1U + V^*S_2V\|$$

so that

$\sup\{\|x+x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \leq \sup\{\|U^*S_1U\| + \|V^*S_2V\|\}$. It is clear that if $A \in NA(H)$ is a contraction with $\mu \in \mathbb{C}$, then

$$\begin{aligned} \|S_1A + AS_2\| &\leq \|(S_1+\mu I)A\| + \|A(S_2-\mu I)\| \\ &\leq \|S_1+\mu I\| + \|S_2-\mu I\| \end{aligned}$$

and therefore

$$\begin{aligned} \sup\{\|U^*S_1U + V^*S_2V\|\} &= \sup\{\|S_1UV^* + UV^*S_2\|\} \\ &\leq \sup\{\|S_1A + AS_2\| : \|X\| \leq 1\} \\ &\leq \min\{\|S_1+\mu I\| + \|S_2-\mu I\| : \mu \in \mathbb{C}\} \end{aligned}$$

It is now sufficient to show that,

$$\min\{\|S_1+\mu I\| + \|S_2-\mu I\|\} \leq \sup\{\|x+x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \quad (4.4.2)$$

If S_1 or S_2 is a scalar operator, the result holds. Suppose none of the two is a scalar. We start with the finite-dimensional case by letting $\|S_1 + \mu_0 I\| + \|S_2 - \mu_0 I\|\| \leq \|S_1 + \mu I\| + \|S_2 - \mu I\| \forall \mu \in \mathbb{C}$. Since $\Gamma(S_1) + \Gamma(S_2) = \Gamma(S_1+\mu I) + \Gamma(S_2-\mu I)$, we may take $\mu_0 = 0$ to simplify the work. By Proposition 4.24, we have unit vectors $\xi, \zeta \in \mathbb{C}^n$ such that $\|S_1\xi\| = \|S_1\|, \|S_2\zeta\| = \|S_2\|$

and $\frac{\langle S_1\xi,\xi\rangle}{\|S_1\|} = \frac{\langle S_2\zeta,\zeta\rangle}{\|S_2\|}$. Let

$x = \begin{pmatrix} \langle S_1\xi,\xi\rangle \\ \{\|S_1\xi\|^2 - |\langle S_1\xi,\xi\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \in \Gamma(S_1), x^{'} = \begin{pmatrix} \langle S_2\zeta,\zeta\rangle \\ \{\|S_2\zeta\|^2 - |\langle S_2\zeta,\zeta\rangle|^2\}^{\frac{1}{2}} \end{pmatrix} \in \Gamma(S_2)$ so that we have $\|x+x^{'}\| = \|x\| + \|x^{'}\| = \|S_1\xi\| + \|S_2\zeta\| = \|S_1\| + \|S_2\|$ as required.

We next look at the infinite-dimensional case. Assume inequality (4.4.2) is false. Then a real number $\varepsilon > 0$ exists which gives $\sup\{\|x+x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} < \|S_1+\mu I\|+\|S_2-\mu I\|-\varepsilon$ for all complex numbers μ. Therefore infinitely many complex numbers $\alpha_1, ..., \alpha_m$ can be found to give $\{\mu \in \mathbb{C} : |\mu| \leq \|S_1\| + \|S_2\|\} \subseteq \bigcup_{i=1}^{m}\{\alpha \in \mathbb{C} : |\alpha - \alpha_i| < \frac{\varepsilon}{4}\}$. Choosing unit vectors $\varsigma_1, ..., \varsigma_m$ and $\eta_1, ..., \eta_m$ in H we have $\|(S_1+\alpha_i I)\varsigma_i\| > \|S_1+\alpha_i I\|-\frac{\varepsilon}{4}$ and $\|(S_2-\alpha_i I)\eta_i\| > \|S_2-\alpha_i I\|-\frac{\varepsilon}{4}$ for all $i = 1, ...m$. Letting H_o be the finite-dimensional subspace of H whose spanning vectors are $\xi_1, ..., \xi_m$, $\eta_1, ..., \eta_m$ and $S_1\xi_1, ..., S_1\xi_m$, $S_2\eta_1, ..., S_2\eta_m$ while S_1', S_2' are compressions of S_1, S_2 respectively with I' also being a compression of I on H_o, we get

$$\min\{\|S_1 + \mu I^{'}\| + \|S_2 - \mu I^{'}\|\}$$

$$= \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \quad (4.4.3)$$

$$\leq \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\} \quad (4.4.4)$$

by applying the finite dimensional case. Moreover, for each $\mu \in \mathbb{C}$ with $|\mu| \leq$

$\|S_1\| + \|S_2\|$, there exists i so that $|\mu - \alpha_i| < \frac{\varepsilon}{4}$ and hence

$$\begin{aligned}\|S_1' + \mu I'\| &> \|S_1' + \alpha_i I'\| - \frac{\varepsilon}{4} \\ &\geq \|(S_1' + \alpha_i I')x_i\| - \frac{\varepsilon}{4} \\ &= \|(S_1 + \alpha_i I)x_i\| - \frac{\varepsilon}{4} \\ &> \|S_1 + \alpha_i I\| - \frac{\varepsilon}{2}\end{aligned}$$

Similarly, $\|S_2' - \mu I'\| > \|S_2 - \alpha_i I\| - \frac{\varepsilon}{2}$ so that
$\|S_1' + \mu I'\| + \|S_2' - \mu I'\| > \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}$. Likewise when $|\mu| > \|S_1\| + \|S_2\|$ we get

$$\begin{aligned}\|S_1' + \mu I'\| + \|S_2' - \mu I'\| &\geq \|2\mu I'\| - \|S_1'\| + \|S_2'\| \\ &> \|S_1\| + \|S_2\| \\ &> \sup\{\|x + x^{'}\| : x \in \Gamma(S_1), x^{'} \in \Gamma(S_2)\}.\end{aligned}$$

Hence this contradicts inequality (4.4.4) which therefore means that inequality (4.4.2) is true. □

Theorem 4.26. *Let $S, T \in NA(H)$ and $\mu_0 \in \mathbb{C}$. Then $\|S - \mu_0 I\| + \|T - \mu_0 I\| \leq \|S - \mu I\| + \| - \mu I T\| \forall \mu \in \mathbb{C}$. Moreover, if $\delta_1 = \|S - \mu_0 I\|$ and $\delta_2 = \|T - \mu_0 I\|$, then*

$$\sup\{\|S - U^*TU\| : U \text{ unitary}\} = \delta_1 + \delta_2 \tag{4.4.5}$$

$$\|g(S) + U^*h(T)U\| \leq \max_{z \in \Gamma(\mu_0;\delta_1)} |g(z)| + \max_{z \in \Gamma(\mu_0;\delta_2)} |h(z)| \tag{4.4.6}$$

for every U and every pair $g(t)$ and $h(t)$ of polynomials.

Proof. Suppose $S, T \in NA(H)$ and that the hypotheses are satisfied by $\delta_1, \delta_2, \mu_o$.

Then by Lemma 4.25 the pair $(S, -T)$ gives

$$\begin{aligned}\sup\{\|S - U^*TU\| : U\text{unitary}\} &= \|S - \mu_0 I\| + \|T - \mu_0 I\| \\ &= \delta_1 + \delta_2\end{aligned}$$

as claimed. Using the von Neumann inequality gives

$$\begin{aligned}\|g(S) + U^*h(T)U\| &\leq g(S) + \|h(T)\| \\ &\leq \max_{z_0 \in \Gamma(\mu_0;\delta_1)} |g(z_0)| + \max_{z_0 \in \Gamma(\mu_0;\delta_2)} |h(z_0)|\end{aligned}$$

□

Proposition 4.27. *Let $S_o, S_1 \in NA(H)$ and $E(S_o, S_1)$ set of complex numbers β_0 satisfying $\|S_o - \beta_0 I\| + \|S_1 - \beta_0 I\| \leq \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in \mathbb{C}$, then $E(S_o, S_1)$ is either a closed line segment or a single point.*

Proof. Being a set of complex numbers, $E(S_o, S_1)$ is clearly a compact set. We now prove the convexity of $E(S_o, S_1)$. Letting $\beta_1, \beta_2 \in E(S_o, S_1)$ and $\beta_0 = t\beta_1 + (1-t)\beta_2$ with $t \in (0,1)$ gives $\|S_o - \beta_0 I\| + \|S_1 - \beta_0 I\| \leq t\{\|S_o - \beta_1 I\| + \|S_1 - \beta_1 I\|\} + (1-t)\{\|S_o - \beta_2 I\| + \|S_1 - \beta_2 I\|\} \leq \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in \mathbb{C}$. Therefore $\beta_0 \in E(S_o, S_1)$. Suppose that $E(S_o, S_1)$ does not include any disk $D_o(\beta_0; \delta) = \{\beta \in \mathbb{C} : |\beta - \beta_0| \leq \delta\}$ with $\delta > 0$. Incase $D_o(\beta_0; \delta) \subseteq E(S_o, S_1)$, we further assume that $\beta_0 = 0$ with (S_o, S_1) in the place of $(S_o - \beta_0 I, S_1 - \beta_0 I)$. Hence

$$\|S_o\| + \|S_1\| = \|S_o - \beta I\| + \|S_1 - \beta I\|\ \forall \beta \in D_o(0; \delta) \tag{4.4.7}$$

Since $D_o(0; \delta) \setminus \{0\}$ is a set which is connected and $g : D_o(0; \delta) \setminus \{0\} \to \mathbb{R}$ is a continuous function defined by $g(\beta) = \|S_o - \beta I\| - \|S_o + \beta I\|$ and given that $-g(\beta) = g(-\beta)$, there must exist $\beta' \neq 0$ to give $g(\beta^{'}) = 0$ that is $\|S_o - \beta' I\| = \|S_o + \beta' I\|$. By equality (4.4.7), it is true that $\|S_1 - \beta' I\| = \|S_1 + \beta' I\|$. But

$$\begin{aligned}
2\|S_o - \beta' I\|^2 &= \|S_o - \beta' I\|^2 + \|S_o + \beta' I\|^2 \\
&\geq \|(S_o - \beta' I)^*(S_o - \beta' I) + (S_o + \beta' I)^*(S_o + \beta^{'} I)\| \\
&= 2\|S_o^* S_o + |\beta'|^2 I\| \\
&= 2\{\|S_o\|^2 + |\beta'|^2\} \\
&> 2\|S_o\|^2
\end{aligned}$$

which leads to $\|S_o - \beta' I\| > \|S_o\|$. Similarly, $\|S_1 - \beta' I\| > \|S_1\|$ and therefore we get $\|S_o\| + \|S_1\| < \|S_o - \beta' I\| + \|S_1 - \beta' I\|$ which contradicts equality (4.4.7). Hence $E(S_o, S_1)$ is either a closed line segment or a single point.

□

Proposition 4.28. *Let $S_1, S_2, S_3 \in NA(H)$ with S_1 and S_2 positive. Then $|\langle S_3 x, y\rangle|^2 \leq \langle S_1 x, x\rangle \langle S_2 y, y\rangle \ \forall x, y \in H$ if and only if $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive in $NA(H \oplus H)$.*

Proof. Assume that $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is a positive operator in $NA(H \oplus H)$. Then $\forall x, y \in H$, the Schwarz inequality for positive operators gives
$\left|\left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle\right|^2$
$\leq \left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle$. Simplification of these inner products gives the required result.

Conversely, suppose the result is true. Then given any $x, y \in H$, we get

$$\begin{aligned}\left\langle \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle S_1x, x\rangle + \langle S_3^*y, x\rangle \\ &\quad + \langle S_3x, y\rangle + \langle S_2y, y\rangle \\ &= \langle S_1x, x\rangle + \langle S_2y, y\rangle + 2Re\langle S_3x, y\rangle \\ &\geq 2\langle S_1x, x\rangle^{\frac{1}{2}}\langle S_2y, y\rangle^{\frac{1}{2}} + 2Re\langle S_3x, y\rangle \\ &\geq 2|\langle S_3x, y\rangle| + 2Re\langle S_3x, y\rangle \\ &\geq 2|\langle S_3x, y\rangle| - 2|\langle S_3x, y\rangle| \\ &= 0\end{aligned}$$

Hence $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive.

□

Lemma 4.29. *Let $S_1, S_2, S_3 \in NA(H)$ and S_1, S_2 be positive with $S_2S_3 = S_3S_1$. If $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \in NA(H \oplus H)$ is positive , then $\begin{pmatrix} g(S_1)^2 & S_3^* \\ S_3 & h(S_1)^2 \end{pmatrix}$ is also positive for continuous non-negative functions g and h on $[0, \infty)$ which satisfies the condition that $g(t)h(t) = t$ for t in the interval $[0, \infty)$.*

Proof. Suppose S_1 and S_2 are both invertible, then for any continuous function h on $[0, \infty)$, $h(S_2)S_3 = S_3h(S_1)$ since $S_2S_3 = S_3S_1$. Similarly, since $t \in [0, \infty)$ implies $g(t)h(t) = t$, then $g(C)h(C) = C$ for any operator $C \in NA(H)$ which is positive. This implies that $h(S_2)S_2^{-\frac{1}{2}}S_3g(S_1)S_1^{-\frac{1}{2}} = S_3$. Therefore,
$\begin{pmatrix} g(S_1)^2 & S_3^* \\ S_3 & h(S_1) \end{pmatrix}$

$= \begin{pmatrix} g(S_1)S_1^{-\frac{1}{2}} & 0 \\ 0 & h(S_2)S_2^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix} \begin{pmatrix} g(S_1)S_1^{-\frac{1}{2}} & 0 \\ 0 & h(S_2)S_2^{-\frac{1}{2}} \end{pmatrix}$ which together with the fact that $\begin{pmatrix} S_1 & S_3^* \\ S_3 & S_2 \end{pmatrix}$ is positive completes the proof. □

Lemma 4.30. *Let $S \in NA(H)$. Then $\begin{pmatrix} |S| & S^* \\ S & |S^*| \end{pmatrix}$ is a positive operator in $NA(H \oplus H)$ where $|S| = (S^*S)^{\frac{1}{2}}$ and $|S^*| = (SS^*)^{\frac{1}{2}}$.*

Proof. On $H \oplus H$, let $A = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$. Then A is self-adjoint and $A^2 = \begin{pmatrix} S^*S & 0 \\ 0 & SS^* \end{pmatrix}$. Since the square root of a positive operator is unique, we get $A = \begin{pmatrix} |S| & 0 \\ 0 & |S^*| \end{pmatrix}$. This therefore means that by the spectral theorem $|A|+|A|$ is positive due to A being self-adjoint. Hence $\begin{pmatrix} |S| & S^* \\ S & |S^*| \end{pmatrix}$ is positive in $NA(H \oplus H)$. □

Theorem 4.31. *Let $S \in NA(H)$ and g and h be as in Lemma 4.29. Then $|\langle S\pi, \pi\rangle| \leq \|g(|S|)\pi\|\|h(|S^*|)y\|$ for each π and y in H.*

Proof. As $S|S|^2 = |S^*|^2 S$, it follows that $S|S| = |S^*|S$ and hence by Lemmas 4.29 and 4.30, we have $\begin{pmatrix} f(|S|)^2 & S^* \\ S & h(|S^*|)^2 \end{pmatrix}$ being positive in $NA(H \oplus H)$. Therefore from Proposition 4.28 the result follows. □

Lemma 4.32. *Let $S_o, S_1 \in NA(H)$. Then $\|S_o+S_1\| = \|S_o\|+\|S_1\|$ is equivalent to $\|S_o\|\|S_1\| \in \overline{W(S_o^*S_1)}$.*

Proof. Let $\|S_o+S_1\| = \|S_o\|+\|S_1\|$. Then a sequence of vectors $\{y_n\}_n$ for each n exists with $\|y_n\| = 1$ such that $\lim_{n\to\infty}\|S_oy_n + S_1y_n\| = \|S_o\| + \|S_1\|$. But

$$\begin{aligned}\|S_oy_n + S_1y_n\| &\leq \|S_oy_n\| + \|S_1y_n\| \\ &\leq \|S_o\|\|y_n\| + \|S_1\|\|y_n\| \\ &\leq \|S_o\| + \|S_1\|\end{aligned}$$

which means that $\lim_{n\to\infty}(\|S_oy_n\| + \|S_1y_n\|) = \|S_o\| + \|S_1\|$. Hence it can be deduced that $\lim_{n\to\infty}\|S_oy_n\| = \|S_o\|$ and $\lim_{n\to\infty}\|S_1y_n\| = \|S_1\|$. Thus the identity
$\|S_oy_n + S_1y_n\|^2 = \|S_oy_n\|^2 + \|S_1y_n\|^2 + 2Re(\langle S_o^*S_1y_n, y_n\rangle)$ shows that
$\lim_{n\to\infty} Re(\langle S_o^*S_1y_n, y_n\rangle) = \|S_o\|\|S_1\|$ and since
$|\langle S_o^*S_1y_n, y_n\rangle| = (Re(\langle S_o^*S_1y_n, y_n\rangle))^2 + Im(\langle S_o^*S_1y_n, y_n\rangle)^2)^{\frac{1}{2}}$ and

$$\begin{aligned}|\langle S_o^*S_1y_n, y_n\rangle| &\leq \|S_o^*S_1y_n\| \\ &\leq \|S_o\|\|S_1\|\end{aligned}$$

then we have $\lim_{n\to\infty}|\langle S_o^*S_1y_n, y_n\rangle| = \|S_o\|\|S_1\|$. Thus $\lim_{n\to\infty} Im(\langle S_o^*S_1y_n, y_n\rangle) = 0$ which implies that $\lim_{n\to\infty}\langle S_o^*S_1y_n, y_n\rangle = \|S_o\|\|S_1\|$ meaning $\|S_o\|\|S_1\| \in \overline{W(S_o^*S_1)}$. Conversely, assume that $\|S_o\|\|S_1\| \in \overline{W(S_o^*S_1)}$ and consider $\{y_n\}_n \in H$, which gives $\lim_{n\to\infty}\langle S_o^*S_1y_n, y_n\rangle = \|S_o\|\|S_1\|$. Then since

$$\begin{aligned}|\langle S_o^*S_1y_n, y_n\rangle| &\leq \|S_oy_n\|\|S_1\| \\ &\leq \|S_o\|\|S_1\|\end{aligned}$$

it follows that $\lim_{n\to\infty}\|S_oy_n\| = \|S_o\|$. Similarly we have $\lim_{n\to\infty}\|S_1y_n\| =$

$\|S_1\|$ and since

$$\|S_o y_n + S_1 y_n\|^2 = \|S_o y_n\|^2 + \|S_1 y_n\|^2 + 2Re(\langle S_o^* S_1 y_n, y_n\rangle)$$

and $\lim_{n\to\infty} Re(\langle S_o^* S_1 y_n, y_n\rangle) = \|S_o\|\|S_1\|$, then $\lim_{n\to\infty}\|S_o y_n + S_1 y_n\| = \|S_o\| + \|S_1\|$. Hence $\|S_o + S_1\| = \|S_o\| + \|S_1\|$. □

Theorem 4.33. *Let $S_o, S_1 \in NA(H)$. Then the statements which follow are equivalent.*

(i). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\|S_o + \beta S_1\| = \|S_o\| + \|S_1\|$

(ii). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\beta\|S_o\|\|S_1\| \in \overline{W(S_1^* S_o)}$

(iii). $\exists\, \beta \in \mathbb{C}$ *with* $|\beta| = 1$ *in order for* $\beta\|S_o\|\|S_1\| \in \sigma_{ap}(S_1^* S_o)$

(iv). $w(S_o^* S_1) = \|S_o^* S_1\| = \|S_o\|\|S_1\|$

(v). $r(S_o^* S_1) = \|S_o^* S_1\| = \|S_o\|\|S_1\|$

Proof. $(i) \Leftrightarrow (ii)$ is as a result of Lemma 4.32. $(iii) \Leftrightarrow (ii)$ since $\sigma_{ap}(S_1^* S_o)$ is contained in $W(S_1^* S_o)$. $(ii) \Leftrightarrow (iv)$ is as a result of $w(S_o^* S_1) \in \overline{W(S_1^* S_o)}$. $(v) \Leftrightarrow (iii)$ since $r(S_o^* S_1) \in \sigma_{ap}(S_1^* S_o)$. $(iv) \Leftrightarrow (v)$ is as a result of the fact that for any operator $A \in B(H)$, $r(A) = \|A\|$ if and only if $w(A) = \|A\|$. □

Proposition 4.34. *Let $S_o, S_1 \in NA(H)$. Then $\|S_o\|\|S_1\| \in \overline{W(S_o^* S_1)}$ and $0 \in \sigma_{ap}(\|S_1\|S_o - \|S_o\|S_1)$ is equivalent to either S_o or S_1 being isometric.*

Proof. Suppose $\|S_o\|\|S_1\| \in W(S_o^* S_1)$. Then we have a sequence $\{y_n\}_{n=1}^{\infty}$ of vectors for all n with $\|y_n\| = 1$ so that $\lim_{n\to\infty}\langle S_o^* S_1 y_n, y_n\rangle = \|S_o\|\|S_1\|$. Therefore

$\lim_{n\to\infty} Re\langle S_o^*S_1y_n, y_n\rangle = \|S_o\|\|S_1\|$ and as

$$\begin{aligned}|\langle S_o^*S_1y_n, y_n\rangle| &\leq \|S_oy_n\|\|S_1y_n\| \\ &\leq \|S_o\|\|S_1\|\|y_n\|^2 \\ &\leq \|S_o\|\|S_1\|\end{aligned}$$

then $\lim_{n\to\infty} \|S_1y_n\| = \|S_1\|$. Similarly, we have $\lim_{n\to\infty} \|S_oy_n\| = \|S_o\|$. But

$$\begin{aligned}\|(\|S_1\|S_oy_n - \|S_o\|S_1y_n)\|^2 &= \|S_1\|^2\|S_oy_n\|^2 + \|S_o\|^2\|S_1y_n\|^2 \\ &\quad -2\|S_o\|\|S_1\|Re(\langle S_o^*S_1y_n, y_n\rangle)\end{aligned}$$

which implies that $\lim_{n\to\infty} \|(\|S_1\|S_oy_n - \|S_o\|S_1y_n)\| = 0$ i.e $0 \in \sigma_{ap}(\|S_1\|S_o - \|S_o\|S_1)$.

For the converse, suppose without loss of generalityS_1 is isometric. Then $0 \in \sigma_{ap}(S_o - \|S_o\|S_1)$ means a sequence $\{y_n\}_n \subseteq H$ exists with $\|y_n\| = 1$ so that $\lim_{n\to\infty} \|S_oy_n - \|S_o\|S_1y_n\| = 0$. From $\|S_oy_n - \|S_o\|S_1y_n\| \geq |\|S_oy_n\| - \|S_o\||$, we have $\lim_{n\to\infty} \|S_oy_n\| = \|S_o\|$. But since $\lim_{n\to\infty}\langle(S_oy_n - \|S_o\|S_1y_n), S_oy_n\rangle = 0$ we deduce that $\lim_{n\to\infty}\langle S_o^*S_1y_n, y_n\rangle = \|S_o\|$ and hence $\|S_o\| \in \overline{W(S_o^*S_1)}$. $\square$

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, conclusions are drawn and recommendations made based on the objectives of the study and the results obtained.

5.2 Conclusion

The study is concluded by pinpointing out the results obtained as per the objectives stated. In chapter one, the introduction was done in which a detailed discussion of the background to the study was made. Some of the basic concepts and definitions were also given in this chapter in addition to some useful results which were crucial to the study. In chapter two, related literature on numerical ranges, spectrum and norms of various operators were reviewed with a survey of the theorems, lemmas and propositions as indicated in the literature.

In chapter three, methods which were used to achieve the stated objectives were highlighted and explained. They included the fundamental inequalities such as the triangle inequality, Cauchy-Schwarz inequality and Bessel's inequality. The parallelogram law and the polarization identity are also other techniques which have been included in chapter three.

The broad scope of this study was to characterize norm-attaining operators in C*-algebras. In chapter four it was shown that the specific objectives outlined in chapter one were achieved and a number of results obtained. The novelty of the study is that numerical ranges and spectra of norm-attaining operators were characterized and their norms established.

For objective one, it was shown that always the numerical range of a norm-attaining operator is non-empty and that zero always belongs to this numerical range as established in Proposition 4.1; The numerical range of a norm-attaining operator is equal to an open unit disc of the set of norm-attaining operators which has been shown in Proposition 4.3 and that it is equal to the convex hull of the eigenvalues of the operator as established in proposition 4.5. In Proposition 4.9, it has been proved that if the numerical range of a norm-attaining operator is closed, then it implies that there exists an extreme point which is contained in its closure.

For objective two, it was established in proposition 4.13 that the convex hull of the spectrum of a norm-attaining operator, S, is equal to the collection of intersections of the closures of the numerical ranges of the operators TST^{-1} for all invertible operators $T \in H$. Theorem 4.16 established that the spectrum of a norm-attaining operator is bounded and that it is closed a subset of the closure of the numerical range of the operator and that if a norm-attaining operator, S, is self-adjoint then its spectrum is real and is contained in the closed interval $[-\|S\|, \|S\|]$ as shown in Theorem 4.18. Lastly, for objective three it was established in Theorem 4.25 that for any norm-attaining operators S_1 and S_2, $\sup \|U^*S_1U + V^*S_2V\| = \min \|S_1 + \mu I\| + \|S_2 - \mu I\|$ where U and V are unitaries and $\mu \in \mathbb{C}$. Also in Lemma 4.32, it was established that $\|S_0 + S_1\| = \|S_o\| + \|S_1\|$ if and only if $\|S_o\|\|S_1\| \in \overline{W(S_o^* S_1)}$. Proposition 4.34 established that for any norm-attaining operators S_o and S_1, if $\|S_o\|\|S_1\| \in W(S_o^* S_1)$ and $0 \in \sigma_{ap}(\|S_1\|S_o - \|S_o\|S_1)$ then this means that either S_o or S_1 is isometric.

5.3 Recommendations

The study focused on characterizing norm-attaining operators in C*-algebras in terms of their numerical ranges, spectra and norms. This alone cannot be said to have fully exhausted the characterization of norm-attaining operators. It may be interesting to narrow down the C*-algebras to commutative C*-algebras and characterize norm-attaining operators in commutative C*-algebras. Therefore further studies can be done to: Characterize the numerical ranges of norm-attaining operators in commutative C*-algebras; characterize spectra of norm-attaining operators in commutative C*-algebras; and determine norms of norm-attaining operators in commutative C*-algebras.

References

[1] **Abramovich Y. A., Alplantis C. D. and Burkinshaw O.,** The Daugavet equation in uniformly convex Banach Spaces, *J. Functional Analysis*, Vol. 97, (1991), 215-230.

[2] **Acosta M. D., Aguirre F. J. and Paya R.,** A new sufficient Condition for Norm attaining Operators, *Rocky mountain J. of math.* Vol. 26, No. 2, (2014), 1-9.

[3] **Acosta M. D.,** Denseness of norm attaining operators into strictly convex spaces, *Proc. Royal Soc. Edinburgh 129 A*, (1999), 1107-1114.

[4] **Acosta M. D. and Galan M. R.,** New characterizations of the Reflexivity in terms of the set of Norm attaining Functionals, *Canadian. Math. Bull.*, Vol. 41 No. 3, (1988), 279-289.

[5] **Agler J.,** Geometric and Topological properties of the Numerical Range, *Indiana Univ. Math. J.* Vol. 31, (1982), 766-767.

[6] **Akemann C. A. and Anderson J.,** *Lyapunov Theorems for Operator Algebras:Memoirs of the American Math. Soc. No. 458*, American Math. Soc., 1991, Providence.

[7] **Akkouchi M.,** Remarks on the Spectrum of bounded Normal operators on Hilbert Spaces, *An. St. Univ. Ovidus Constanta*, Vol. 16, No. 2, (2008), 7-14.

[8] **Alaminos J., Choi Y. S., Kim S. G. and Paya R.,** Norm attaining bilinear forms on spaces of continuous functions,*Glasgow Math. J.*, Vol. 40 (1998), 359-365.

[9] **Alaminos J., Paya R. and Villena A.,** Norm attaining bilinear forms on C^*-algebras, *Studia Mathematica*, Vol. 157, No. 1, (2003), 47-56.

[10] **Albeveiro S., Motovilov A. K. and Tretter C.,**, Bounds on the Spectrum and Reducing Subspaces of a J-Self-adjoint Operator, *Indiana Univ. Math. J.*, Vol. 59, No. 5, (2010), 1737-1776.

[11] **Ali I., Yang H. and Shakor A.,** Generalized Norms inequalities for absolute value Operators, *Int. J. of Anal. and Appl.*, Vol. 5 No. 1, (2014), 1-9.

[12] **Alois K., Oldřich J. and Svatopluk F.,** Function Spaces, *Noordoff Int. Publishing, Leyden*, (1977).

[13] **Ando T. and Li C. K.,** (Special edition), The Numerical Range and Numerical Radius, *Linear and Multilinear Algebra*, Vol. 37, No. 1, (1994).

[14] **Arveson W.,** $C*$-algebras and numerical linear algebra, *J. Funct. Anal.*, Vol. 122, No. 2, (1994), 333-360.

[15] **Arveson W.,** Improper filtrations for C^*-algebras: Spectra of unilateral tridiagonal Operators, *Acta. Sci. Math. (Szeged)*, Vol. 57, No. 1-4, (1993), 11-24.

[16] **Arveson W.,** *An Invitation to C^*-Algebras*, Springer-Verlag, New York, 1976.

[17] **Arveson W.,** Subalgebra of C^*-algebras II, *Acta. Math.*, Vol. 128, (1972), 271-308.

[18] **Atkinson F. V., Langer H., Mennicken R. and Shkalikov A. A.,** The essential Spectrum of some Matrix Operators, *Math. Nachrichten*, Vol. 167, (1994), 5-20.

[19] **Axelsson O., Lu H. and Polman B.,** On the numerical radius of matrices and its application for iterative solution methods, *Linear and Multilinear Algebra*, Vol. 37, (1994), 225-238.

[20] **Bauer F. L.,** On the field of values subordinate to a norm, *Num. Math.*, Vol. 4, (1962), 103-111.

[21] **Bauer F. L. and Fike C.T.,** Norms and exclusion Theorems, *Num. Math.*, Vol. 2, (1960), 137-141.

[22] **Bauer F. L., Ster J. and Witzgall C.**, Absolute norms and monotonic norms, *Num. Math.*, Vol. 3, (1961), 257-264.

[23] **Bayasgalan T.,** The numerical range of linear operators in spaces with an indefinite metric (Russian), *Acta Math. Hungar.*, Vol. 57, (1991), 157-168.

[24] **Beauzamy B.,** *Introduction to Banach Spaces and their Geometry*, North-Holland Pub. Company, New York, 1982.

[25] **Berg I. D.,** On Norm Approximation of functions of Operators in the Calkin Algebra, *Pro. of the Irish Royal Acad. Sec.*, Vol. 78, (1978), 143-148.

[26] **Berger C. A. and Stampfli J. G.,** Mapping theorems for the numerical range, *American. J. Math.*, Vol. 89, (1967), 1047-1055.

[27] **Bhatia R.,** Perturbation inequalities for the absolute map in norm ideals of operators, *J. of Operator Theory*, Vol. 19, (1988), 129-136.

[28] **Binding P., Farenick D. R. and Li C. k.,** A dilation and norm in several variable operator theory, *Canadian. J. Math.*. Vol. 47, (1995), 449-461.

[29] **Bishop E. and Phelps R.R.,** The support functionals of a convex set, proceedings of the Symposium of Pure Math. VII, (Convexity), *American. Math. Soc.*, (1963), 27-35.

[30] **Bishop E. and Phelps R.R.,** A proof that every Banach space is subreflexive, *Bulletin of American Math. Soc.*, Vol. 67, (1961), 97-98.

[31] **Bishop E. and Phelps R.R.,** Subreflexive normed linear spaces, *Arch. Math.*, Vol. 8, (1957), 444-450.

[32] **Bohnenblust H. F. and Karlin S.**, *Contributions to the Theory of Games*, Princeton University Press, Princeton, (1950).

[33] **Bollobás B.,** An extension to the Theorem of Bishop-Phelps, *Bulletin of London Math. Soc.*, Vol. 2, (1970), 181-182.

[34] **Bonsall F. F. and Duncan J.,** *Numerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras*, Cambridge University Press, Cambridge, 1971.

[35] **Bonsall F. F., Duncan J.,** *Numerical Ranges, Vol.I*, Cambridge University Press, Cambridge, 1971.

[36] **Bonsall F. F. and Duncan J.,** Studies in functional analysis-numerical ranges, Studies in Mathematics, *The Math. Ass. of America*, Vol.21, (1980).

[37] **Bӧttcher A., Chithra A.V and Namboodiri M. N. N.,** Approximation of approximation numbers by truncation, *Integr. equ. oper. theory*, Vol. 39, (2001), 387-395.

[38] **Bouldin R.,** Essential spectrum for a Hilbert space operator, *Transactions of the American Math. Soc.*, Vol. 163, (1972), 437-445.

[39] **Bourgain J.,** On dentability and the Bishop-Phelps property, *Israel J. of Math.*, Vol. 28, (1977), 265-271.

[40] **Brown A. and Pearcy C.,** *Introduction to operator Theory: Elememts of Functional analysis*, Springer Science and Business media, New York, 2012.

[41] **Brown J. R.,** Approximation theorems for Markov Operators, *Pacific J. of Math.*, Vol. 16, (1966), 13-23.

[42] **Ching-Hwa M.,** On the numerical range of an operator, *Proceedings of the American Math. Soc.*, Vol. 14, (1963), 167-171.

[43] **Chi-Kwong L. and Yiu-Tung P.,** Spectrum, numerical range and Davis-Wielandt Shell of a normed operator, *Glasgow Math. J.*, Vol. 51, (2009), 91-100.

[44] **Chien M. T. and Tam B. S.,** Circularity of the numerical range, *Linear Algebra and Appl.*, Vol. 201, (1994), 113-133.

[45] **Choi M. D. and Li C. K.,** The ultimate estimate of the upper norm bound for the summation of operators, *J. of Functional Anal.*, Vol. 232, (2006), 455-476.

[46] **Choi Y. S.,** Norm attaining Bilinear forms on $L_1[0,1]$, *J. of Math. Anal. and Appl.*, Vol. 211, (1997), 295-300.

[47] **Choi Y. S., Lee H. J. and Song H. G.,** Denseness of norm attaining mappings on Banach Spaces, *Publication of RIMS Kyoto Univ.*, Vol. 46, (2010), 171-182.

[48] **Christopher H.,** Banach and Hilbert Space review, *Lecture Notes*, (2006).

[49] **Conway J. B,** *A Course in Functional Analysis, 2nd edition*, Springer-Verlag, New York, (1990).

[50] **Crouzeix M.,** Numerical range and functional calculus in Hilbert space, *J. Functional Anal.*, Vol. 242, No. 2, (2007), 658-668.

[51] **Damak M. and Jeribi A.,** On the essential Spectra of matrix operators and applications, *Electronic J. of Differential Equations*, Vol. 2007, No. 11, (2007), 1-16.

[52] **Damir B.,** Compact operators, the essential spectrum and the essential numerical range, *Math. communications*, Vol.3, (1998), 103-108.

[53] **Davies B. E.,** Linear Operators and their Spetra , *Michigan Math. J.*, Vol. 4, issue 3, (1957), 261-263.

[54] **Davis W. J. and Phelps R. R.,** The Radon-Nikodym property and dentable sets in Banach Spaces, *Proceedings of American. Math. Soc.*, Vol. 45, (1974), 119-122.

[55] **De Barra G., Giles J. R. and Sims B.,** On the numerical range of compact operators, *J. London Math. Soc.*, Vol. 2, No. 5, (1972), 704-706.

[56] **Diestel J.,** *Geometry of Banach Spaces-Selected Topics*, Springer, New York, 1975.

[57] **Donoghue W. F.,** On the numerical range of a bounded Operator , *cambridge Univ. Press, New York*, 2007.

[58] **Douglas B. and Robin H.,** A constructive analysis that the numerical range is convex, *J. of London Math. Soc.*, Vol. 3, (2000), 191-206.

[59] **Dunford N.,** Spectral Operators, *Pacific J. Math.*, Vol. 4, (1954), 321-354.

[60] **Enders A. R.,** A historical Perspective of Spectrum Estimation, *Proceedings of the IEEE*, Vol. 70, No.9, (1982), 885-907.

[61] **Enflo P., Kover J. and Smithies L.,** Denseness for norm-attaining operator-valued functions, *Linear Algebra and its Appl.*, Vol. 338, (2001), 139-144.

[62] **Embry M. R.,** Similarities involving normal operators on Hilbert space, *Pacific J. of Math.*, Vol. 35, (1970), 331-335.

[63] **Embry M. R.,** The numerical range of an operator, *Pacific J. of Math.*, Vol. 32, (1970), 647-650.

[64] **Farenick D.,** matricial Extensions of the numerical range:A brief survey, *Linear and Multilinear Algebra*, Vol. 34, (1993), 197-211.

[65] **Fillmore P. A., Stampfli J. G. and William J. P.,** On the essential numerical range, the essential Spectrum and a Proplem of Halmos, *Acta. Sci. Math.(szeged)*, Vol. 33, (1972), 179-192.

[66] **Finet C. and Paya R.,** Norm attaining operators from L_1 into L_∞, *Israel J. Math.*, Vol. 108, 139-143.

[67] **Furuta T. and Nakamoto R.,** On the numerical range of an operator, *Proceedings of Japan Acad.*, Vol. 37, No. 3, (1971), 279-283.

[68] **Gang C. and Yun F.,** On the connected components of the spectrum of the extended character ring of a finite group, *J. of Algebra*, Vol. 312, (2007), 689-698.

[69] **Gau H. L. and Wu P. Y.,** Condition for the numerical range to contain an elliptic disc, *Linear Algebra Appl.*, Vol. 364, (2003), 213-222.

[70] **Gau H. L. and Wu P. Y.,** Line segments and elliptic arcs on the boundary of a numerical range, *Linear and Multilinear Algebra*, Vol. 56, (2008), 131-142.

[71] **Gaur A. K. and Kovarik Z. V.,** Norms on unitizations of Banach algebras, *Proceedings of American Math. Soc.*, Vol. 117, No. 1, (1993), 111-113.

[72] **Gowers W.,** Symmetric block bases of sequences with large average growth, *Israel J. Math.*, Vol. 69, (1990), 129-149.

[73] **Gustafson K. E. and Rao D. K. M.,** *Numerical Range. The field of values of Linear Operators and Matrices*, Springer-Verlag, New York, 1997.

[74] **Halmos P. R.,** Normal dilations and extensions of operators, *Summa Brasil. Math.* Vol. 2, (1950), 125-134.

[75] **Halmos P. R.,** *A Hilbert space Problem Book*, Van Nostrand, 1967.

[76] **Halmos P. R.,** *A Hilbert space Problem Book, Second Ed.*, Springer-Verlag, New York, 1982.

[77] **Hansen A. C.,** On the approximation of spectra of linear operators on Hilbert spaces, *J. of Functional Anal.*, Vol. 254, (2008), 2092-2126.

[78] **Hausdorff F.,** Der Wertevorrat einer Bilinearform, *Math. Z.,*, Vol. 3, (1919), 314-316.

[79] **Harris L. A.,** The numerical range of functions and best approximation, *Proceedings of Cambridge Philos. Soc.*, Vol. 76, (1974), 133-141.

[80] **Hladnik, Milan and Matjaz O.,** Spectrum of the product of operators, *Proceedings of American Math. Soc.*, Vol. 102, No. 2, (1988), 300-302.

[81] **Horn R. A., Johnson C. R.,** *Topics in Matrix Analysis*, Cambridge Univ. Press, cambridge, 1991.

[82] **Hwa-Long G. and Pei Y. W.,** Excursions in Numerical Ranges, *Bulletin of the Inst. of Math. Academia Sinica (New series)*, Vol. 9, No. 3, (2014), 351-370.

[83] **Istratescu B.,** *Introduction to Linear operator Theory*, Marcel Dekker, New York, 1981.

[84] **Itoh T.,** The Haagerup type cross norm on $C*$-algebras, *Proceedings of the American Math. Soc.*, Vol. 109, (1990), 689-695.

[85] **Iwanik A.,** Norm attaining operators on Lebesgue Spaces, *Pacific J. of Math.*, Vol. 83, (1979), 381-386.

[86] **Jahedi K. and Yousefi B.,** Numerical range of operators acting on Banach spaces, *J. Czech Math. 62:495.doi:10.1007/s 10587-012-0024-7*, (2012), 495-503.

[87] **Johnson J. and Wolfe J.,** Norm attaining operators, *Studia Math.*, Vol. 65, (1980), 7-19.

[88] **Johnson J. and Wolfe J.,** Norm attaining operators and simultaneously Continuous Retractions, *Proceedings of American Math. Soc.*, Vol. 86, No. 4, (1982), 609-612.

[89] **Jordan B.,** The spectrum of a self-adjoint operator is a compact subset of $\mathbb{R}$, *Lecture notes, University of Toronto*, (2014).

[90] **Keeler D. S., Rodman L. and Spitkovisky I. M.,** The numerical range of 3×3 matrices, *Linear Algebra Appl.*, Vol. 252, (1997), 115-139.

[91] **Khalaghai J. M.,** *Operator Equations in Hilbert Spaces, PhD Thesis*, Univ. of Nairobi, 1983.

[92] **Khalaghai J. M. and Otieno M. O. E.,** On quasiaffinity and quasi-affine inverses of partial isometries, *Bulletin of the Allahabad Math. Soc.*, Vol. 15, (2000), 35-40.

[93] **Khalid S. and Albadawi H.,** Norm inequalities for the absolute value of Hilbert Space operators, *J of Linear and multilinear Algebra*, Vol. 58, No. 4, (2010), 453-463.

[94] **Kingangi D. N.,** On the numerical ranges, *Int. J. of Scientific and Innovative Math. Research*, Vol. 3, No. 8, (2015), 31-36.

[95] **Kittaneh F.,** A numerical radius inequality for certain operator sums, *Studia Math.*, Vol. 158, (2003), 17.

[96] **Kittaneh F.,** Notes on some inequalities for Hilbert space operators, *Publ. RMS Kyoto Univ.*, Vol. 24, (1988), 283-293.

[97] **Klein E. M.,** The numerical range of a toeplitz operator, *Proceedings of the American Math. Soc.*, Vol. 35, No. 1, (1972), 101-103.

[98] **Kumar K. and Sinclair A. M.,** Equivalence of norms on operator space tensor products of $C*$-algebras, *Transactions of the American Math. Soc.*, Vol. 350, No. 5, (1998).

[99] **Kumar K., Namboodiri M. N. N. and Serra-Capizzano S.,** Perturbation of operators and approximation of spectrum, *Proceedings of Indian Acad. Sci. (Math. Sci.)*, Vol. 124, No. 2, (2014), 205-224.

[100] **Lancaster J. S.,** The boundary of the numerical range, *Proceedings of the American Math. Soc.*, Vol. 49, No. 2, (1975), 393-398.

[101] **Lancaster J. S.,** Norms on direct sums and tensor products, *J. Mathematics of computation*, Vol. 26, (1972), 401-414.

[102] **Lang S.,** *Linear algebra*, Springer-Verlag, New York, 1987.

[103] **Laura B.,** Noncontinuity of spectrum for the adjoint of an operator, *Proceedings of the American Math. Soc.*, Vol. 128, No. 1, (1999), 173-182.

[104] **Lee J. I.,** On the norm attaining operators, *Korean J. Math.*, Vol. 20, (2012), 485-491.

[105] **Leif M.,** Hilbert spaces and operators on Hilbert spaces, Ventures Publishing APS, (2009).

[106] **Li C. K. and Tsing N. K.,** C-numerical ranges and C-numerical radii, *Linear and Multilinear Alg.*, Vol. 37, (1994), 51-52.

[107] **Lin C. S.,** On operator whose norm is an eigenvalue, *Bulletin of the institute of Math. Academica Sinica*, Vol. 31, No. 1, (2003), 1-9.

[108] **Lindenstrauss J.,** On extreme points in l_1, *Israel J. Math.*, Vol. 12, (1966), 59-61.

[109] **Lindenstrauss J.,** On operators which attain their norm, *Israel J. Math.*, Vol.1, (1963), 139-148.

[110] **Lumer G.,** Semi-inner product spaces, *Transactions of the American Math. Soc.*, Vol. 100, (1961), 29-43.

[111] **Marcus M. and Pesc C.,** Computer generated numerical range and some resulting theorems, *Linear and Multilinear Alg.*, Vol. 20, (1987), 121-147.

[112] **Mathieu M.,** Properties of the product of two derivations of a C^*-algebra, *Canadian Math. Bulletin*, Vol. 32, (1989), 490-497.

[113] **Maynard H. B.,** A geometrical Characterization of Banach spaces with the Radon Nikodym property, *Transactions of the American Math. Soc.*, Vol. 185, (1973), 493-500.

[114] **Mecheri S.,** The numerical range of linear operators, *Notes*, 1991.

[115] **Miguel M.,** Lecture Notes, *Univ. of Granada*, 2009.

[116] **Miguel M.,** Norm attaining compact operators, *Israel J. Math.*, Vol. 43, (1982), 273-276.

[117] **Miguel M., Javier M. and Paya R.,** On the Intrinsic and the Spatial numerical range, *J. Math. Anal and Appl.*, Vol. 318, (2006), 175-189.

[118] **Mitrinovic D. S., Pecaric J. E. and Fink A. M.,** *Classical and new inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[119] **Morenz P. B.,** *The structure of C^*-Convex sets, PhD Thesis*, Univ. of Toronto, 1992.

[120] **Murphy G. J,** C*-algebras and operator theory, *Academic Press Inc., San Diego*, 1990.

[121] **Okelo N. B.,** Norm-attainability and Range-Kernel orthogonality of elementary operators, *Communications in Advanced Math. Sciences.*, Vol.1, no.2, (2018), 1-9.

[122] **Okelo N. B.,** Norms and Norm-attainbility of normal operators and their applications, *The 2015 JKUAT Scientific conference.*, (2017), 83-85.

[123] **Okelo N. B.,** On Dvoretsky's Theorem and Norms of Elementary Operators, *Int. J. of Pure and Appl. Science and Technology*, Vol.2, no.2, (2011), 46-53.

[124] **Okelo N. B., Agure J.O., Ambogo D.O.,** Norms of Elementary operators and characterization of Norm-Attainable operators, *Int. J. Math. Anal.*, Vol. 4, (2010), 1197-1204.

[125] **Osikiewicz B., Tonge A.,** An interpolation approach to Hardy-Littlewood inequalities for norms of operators on sequence spaces, *Linear Algebra and its Applications*, Vol.331,(2001), 1-9.

[126] **Ostrowski A. and Scheneider H.,** Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.*, Vol. 4, (1962), 72-84.

[127] **Partington J. R,** Norm attaining Operators, *J.Israel J.Math*, Vol. 43, (1982), 273-276.

[128] **Pandey S. K. and Vern I. P.,** A spectral characterization of AN Operators, *J. Australian. Math. Soc.*, Vol. 102, (2017), 369-391.

[129] **Paya R. and Saleh Y.,** Norm attaining Operators from $L_1(\mu)$ to $L_\infty(V)$, *Arch. Math.*, Vol. 75, (2000), 380-388.

[130] **Paya R. and Finet C.,** Norm attaining Operators from L_1 into L_∞, *Israel J. Math.*, Vol. 108, (1998), 139-143.

[131] **Pellegrino D. and Teixeira E. V.,** Norm optimization problem for linear Operators in classical Banach Spaces, *Bulletin of the Brazilian Math. Soc., New Series*, Vol. 40, No. 3, (2009), 417-431.

[132] **Radjabalipour M. and Radjavi H.,** On the Geometry of Numerical Range, *Pacific J. of Math.*, Vol. 61, (1975), 507-511.

[133] **Ralph H. and Anton R. S.,** Norms of positive operators on L^p spaces, *Proceedings of the American Math. Soc.*, Vol. 109, No. 1, (1990), 135-146.

[134] **Rieffel M. A.,** Dentable subsets of Banach spaces, with applications to Radon-Nikodym theorem, *Proceedings Conf. Functional Anal.*, (1967), 71-77.

[135] **Russel G. B and Lewis P. W.,** Norm attaining operators and Norming functionals, *Proceedings of the American Math. Soc.*, Vol. 85, No. 2, (1982), 245-250.

[136] **Schachermayer W.,** Norm attaining operators on some classical Banach spaces, *Pacific J. of Math.*, Vol. 105, (1983), 427-438.

[137] **Schachermayer W.,** Norm attaining operators and renormings of Banach Spaces, *Israel J. Math.*,Vol. 44, (1983), 201-212.

[138] **Scheneider H. and Strang W. G.,** Comparison theorems for Supremum norms, *Num. Math.*, Vol. 4, (1962), 15-20.

[139] **Schnaubelt R.,** Functional Analysis, *Class Notes, Karlsruhe*, (2010).

[140] **Sevilla J. M. and Paya R.,** Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces, studia Mathematica, Vol. 127, (1998), 99-112.

[141] **Shapiro J. H.,** Notes on the numerical range, *Lecture notes*, 2004.

[142] **Shapiro J. H.,** A survey of canonical forms and invariants for unitary similarity, *Linear Algebra Appl.*, Vol. 147, (1991), 101-167.

[143] **Shebrawi K. and Albadawi H.,** Norm inequalities for the absolute value of Hilbert space operators, *Linear and Multilinear Alg.*, Vol. 58, (2010), 453-463.

[144] **Shifang Z., Zhenying W. and Huaijie Z.,** Continuous spectrum, point spectrum and residual spectrum of operator matrices, *Linear Algebra and its Appl.*, Vol. 433, (2010), 653-661.

[145] **Sims B.,** On a connection between the numerical range and spectrum of an operator on a Hilbert space, *J. London Math. Soc.*, Vol. 2, No. 8, (1974), 57-59.

[146] **Sims B.,** On numerical range and its application to Banach algebras, *Bulletin of the Australian Math. Soc.*, Vol. 7, (1972), 307-308.

[147] **Skoufranis P.,** Numerical ranges of operators, *Lecture Notes.*, 2014.

[148] **Stampfli J. G.,** The norm of a derivation, *Pacific J. of Math.*, Vol. 33, No. 3, (1970), 737-747.

[149] **Stone M. H.,** Linear transformations in Hilbert space, *American Math. Soc. Coll. Publ.*, Vol. 15, (1982), 1080-1092.

[150] **Tam B. S.,** Circularity of numerical ranges and block-shift matrices, *Linear and Multilinear Algebra*, Vol. 37, (1994), 93-109.

[151] **Tam B. S. and Yang B. S.,** On matrices whose numerical ranges have circular or weak circular symmetry, *Linear Algebra and its Appl.* , (1999), 193-221.

[152] **Timoney R. M.,** Norms and CB-norms of Jordan elementary operators, *Bulletin of Science Math.*, Vol. 127, (2003), 597-609.

[153] **Timoney R. M.,** Computing the Norms of Elementary operators, *Illinois J. of Math.*, Vol. 47, (2003), 1207-1226.

[154] **Timoney R. M.,** Norms of elementary operators, *Irish Math. Soc. Bulletin*, Vol. 46, (2001), 13-17.

[155] **Toeplitz O..,** Das algebraische Analogon zu einem satze von Fejer, *Math. Z.,* Vol.2 (1918), 187-197.

[156] **Tsa M. C. and Wu P. Y.,** Numerical ranges of weighted shift matrices, *Linear Algebra and its Appl.*, Vol. 435, (2011), 243-254.

[157] **Tso, Shu-hsien and Pei Y.W.,** Matricial Ranges of Quadratic Operators, *The Rocky Mountain J. of Math.*, Vol. 29, No. 3, (1999), 1139-1152.

[158] **Uhl J. J. Jr.,** Norm attaining opertaors on $L^1[0,1]$ and the Radon-Nikodym property, *Pacific J. Math.*, Vol. 63, (1976), 293-300.

[159] **Weidmann J.,** *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York, 1980.

[160] **Williams J. P.,** The Numerical Range and the Essential Numerical Range, *Proceedings of the American Math. Soc.*, Vol. 66, No. 1, (1977), 185-186.

[161] **Williams J. P.,** Finite operators, *Proceedings of American Math. Soc.*, Vol. 21, (1970), 129-136.

[162] **Williams J. P., Crimmins T.,** On the numerical radius of a linear Operator, *Amer. Math. Monthly.*, Vol. 74, (1967), 832-833.

[163] **Williams J. P.,** Spectra of products and numerical ranges, *J. of Math. Anal. and Appl.*, Vol. 17, (1967), 214-220.

[164] **Wu P. Y.,** Numerical ranges as circular discs, *Applied Mathematics Letters*, Vol. 24, (2011), 2115-2117.

[165] **Yuming S., Chunmei S. and Guojing R.,** Spectral properties of self-adjoint subspaces, *J. of Linear Algebra and its Appl.*, Vol. 438, (2013), 191-218.

[166] **Zhang F.,** *Matrix Theory, Basic Results and Techniques, Second Edition*, New York, 2010.